# Avoiding negative numbers and complex numbers thanks to the study of the geometrical nature of some arithmetical and polynomial problems

Jaime Vladimir Torres-Heredia Julca*
torres.heredia@concept-global.net

August 30, 2025

**Abstract**

In this paper we will see that we can avoid the concepts of negative number and complex number thanks to the study of the underlying vector nature of some arithmetic and polynomial problems. We will see that the geometrical models used until now to represent negative numbers and complex numbers and their operations are not just interpretations or models. Translations, rotations and homotheties are what we need to solve several problems. We will see that what we call "negative numbers" and "complex numbers" are just the solutions of vector calculations and equations. All that is the consequence of the fact that geometrical considerations are unavoidable when we think about debts and gains and when we try to solve some polynomial equations. We will see that thanks to the solutions of those vector equations we can construct paths in the plane. We will also give the vector meaning of the formulas of De Moivre and Euler. An interpretation of the vertical axis linked to gains and losses will also be given.

## 1 Introduction

The use of negative numbers began in China and India many ceturies ago and they were used in Europe from the Middle Age (see [Smith2001] ). The study of the equations in Europe lead to the apparition of complex numbers because of the square root of negative numbers needed to solve some equations. Raffael Bombelli constructed those numbers in 1572 (see [Katz2009], p 404 ). Between the end of the 18th and the beginning of the 19th century, several mathematicians as Jean-Robert Argand or Gauss found geometrical interpretations of complex numbers. But, as for negative numbers, many mathematicians were not satisfied with "imaginary numbers" even if there are geometrical interpretations. In this paper we will see that actually there are no imaginary numbers. We will see that several arithmetical and polynomial problems are in fact geometrical problems which lead to solutions that are geometrical operations. Those geometrical operations have been considered as complex number until now.

## 2 Representing gains, losses and debts in Ancient China and India

They used rods with two colors. Red rods for gains and black for losses. We can see the descriptions of the calculations in the "The Nine Chapters on the Mathematical Art" commented by Liu Hui

*Jaime V. Torres is an independent researcher. He studied philosophy and computer science for the humanities at the University of Geneva, and holds a M.A. He is also a self-taught mathematician.

(see [Kangshen2000]). There were rules for addition and substraction as the nowadays rules (see see [Mumford]).

They counted 1 unit of gain, two units of gain, etc. And 1 unit of loss, 2 units of losses, etc..

Jean-Claude Martzloff explains that those negative numbers were used only as computational intermediates (see [Martzloff2006], p. 200).

He explains also that "the Chinese from antiquity had a predilection for an analysis of all sorts of phenomena in terms of complementary couples, positive or negative" (see [Martzloff2006], p. 200). Symmetries were also important for them.

In India negative numbers were used to represent debts (see [Smith2001], [Mattessich1998] and [Brahmagupta]).

## 3 The geometrical nature of losses and gains

So there was an underlying idea of symmetry in China. There were losses, a state of equilibirum, and gains. That is geometrical.

There was a geometrical underlying structure of negative numbers, but, as far as we know, Indians and Chinese were not aware of that.

It is now clear that there is a link between negative numbers and the number line thanks to Descartes' representation of numbers. But in the number line, negative numbers are just representations. They are positions in the number lines, More precisely, they are labels of points in the number line.

"-5" is a label, for example. It's not a really a number. We should talk about units of debt, for example: 1 unit of debt, 2 units of debt. etc. Those quantities of losses or gains car be represented as in figure 1:

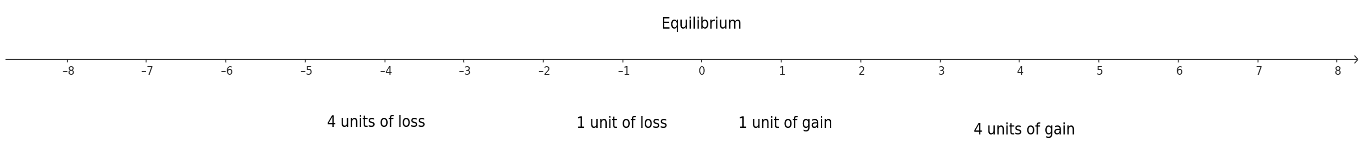


Figure 1: Representation of the number of units of gains and losses

So there are no really negative numbers. We count, with natural numbers, the number of units of debts, or the numbers of units of length from the middle of the line to the left. And we use labels as "-5" to indicate a position on the line. We use it also for counting the units of debts. Obviously, we can also use magnitudes on the line. For exemple "-4.25" represents also a point on the line between the points represented by "-5" and "-4".

There has been studies about it made by cognitive scientists (see [LakoffNúñez2000]) who recall the underlying geometrical nature of negative numbers and the importance of that fact for education.

So, that is why the complex numbers are linked to geometrical operations. In fact there are no complex numbers as we will see below. What we have are just geometrical operations who appear because of the use of negative labels when we try to solve certain polynomial equations.

# 4 Working with negative labels and vectors

During the last 2 centuries, mathematicians have developped the techniques of the euclidean vector space. Those techniques with vectors have integrated the use of negative numbers with their different operations. We are going to use those techniques to show the resolution of some problems, but, as we saw above, instead of negative numbers we will talk about negative labels.

Obviously, the aim of the presentation of the vector techniques is to show the underlying nature of calculations for debts, gains, etc. It's clear that, for practical reasons, we can always work with negative and positive labels, as "-4" or "+5", with the well known arithmetical rules.

For elementary problems as "John has 5 apples and looses 3. How many apples does John has?" we have the following calculation, which corresponds to quantities of material objects:

$$5 - 3 = 2$$

For those calculations we use natural numbers. And with them we can substract a smaller number from a bigger number.

A problem arise when we want to perform, for example, the following calculation:

$$150 - 200 =?$$

This calculation correponds, for example, to the following problem: John has \$150 and buys a machine for \$200. What is his financial situation after that? This problem has no solution with natural numbers. In fact John borrows from a bank \$50 to buy the machine. After that, he is in a financial situation of indebtedness with respect to the bank. And it in his bank account it is usually represented by "-50".

As we saw below "negative numbers" were introduced to perform calculations about debts. And John passed from a state of equilibrium with respect to the bank to state of indebtedness. And the mathematical description of the situation is intrinsically geometric.

So that problem can not be solved with natural numbers. In fact it is a geometrical problem. And we will solve it with the techniques of the euclidean vector space. Instead of the calculation above, we will perform the following calculation:

$$\begin{pmatrix} 150 \\ 0 \end{pmatrix} - \begin{pmatrix} 200 \\ 0 \end{pmatrix}$$

And this is the important point: we pass from a calculation involving natural numbers to a calculation involving points and vectors.

The calculation

$$150 - 200$$

is a schema and the vector equation will follow a part of his structure.

So, we pass from an arithmetical problem to a geometrical problem.

We will see more details about this calculation. The fact the he has, at the begining,\$150 is represented by a point in the line:

$$A = (150, 0)$$

Then he spends \$150 and \$50 (from the bank). So, first, we work now with a translation of $A$ of 150 units towards the left side. For this translation T, we use a vector $\vec{v}$. In that vector, the component "-5" is a label, as we saw above:

$$\vec{OA} = \begin{pmatrix} 150 \\ 0 \end{pmatrix}$$
$$\vec{v} = \begin{pmatrix} -150 \\ 0 \end{pmatrix}$$

So, in order to find $T_{\vec{v}}(A)$ we calculate:

$$\begin{aligned} \vec{OA} + \vec{v} &= \begin{pmatrix} 150 \\ 0 \end{pmatrix} + \begin{pmatrix} -150 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So

$$T_{\vec{v}}(A) = (0, 0)$$

After that, he spends \$50 from the bank. We use a new translation towards the left:

$$\vec{w} = \begin{pmatrix} -50 \\ 0 \end{pmatrix}$$

So, in order to find $T_{\vec{w}}((0, 0))$ we calculate::

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \vec{w} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -50 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -50 \\ 0 \end{pmatrix} \end{aligned}$$

We arrive at a point B on the line

$$B = (-50, 0)$$

So finally he has a debt of \$50. As we have seen before, he actually has 50 units of debt and it is represented also by the label "-50".

We have seen some arithmetical operations. For the multiplications, sometimes we have to perform the calculation:

$$3 \times (-4)$$

Obviously, there is no solution with natural numbers. It will become a vector calculation. "3" represents a point in the number line. Its distance to the origin is 3. We want to repeat or reproduce 4 times that distance, but in the opposit direction. This the description of an homothety H. The label "-4" will be used as a scalar and we will have to perform the following vectorial calculation:

$$C = (3, 0)$$

$$\vec{OC} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\lambda = -4$$

$$\begin{aligned} \vec{OC}\lambda &= \begin{pmatrix} 3 \\ 0 \end{pmatrix} (-4) \\ &= \begin{pmatrix} -12 \\ 0 \end{pmatrix} \end{aligned}$$

We arrive at a point D on the number line:

$$D = (-12, 0)$$

So

$$H_\lambda(C) = (-12, 0)$$

But that result can be obtained also by performing a rotation of center O and an homothety with $\lambda = 4$. This will be a rotation of 180° (see [LakoffNúñez2000]). So we have to calculate:

$$D = R(180°) \circ H_4(C)$$

## 5 Solving first-degree equations without natural solutions

Many equations can be solved using natural numbers. For example:

$$2x + 4 = 10$$

Its solution is $x = 3$.

But if we have the following equation:

$$10 + 3x = 4$$

we see that we will have to solve $3x = 4 - 10$. But $4 - 10$ can not be calculated with natural numbers.

So, with the schema of the first equation, we will have a vector equation. And the question will be:

*Starting from the point (10,0), which geometrical operation will lead to the point (4,0)?*

So the solution won't be a number, but a geometrical operation.

As the addition appears in the first equation, we see that the operation involves a translation. We also see that there is a factor 3, so the vector of the translation will undergo a homothety $H_3$.

So we get the following vectorial equation:

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

So, after the vectorial operations we get:

$$\vec{x} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

And the finally the solution is the following translation:

$$T_{H_3(\vec{x})}$$

So, to reach the point (4,0) from the point (10,0) we must to perform:

$$T_{H_3(\vec{x})}((10,0))$$

# 6 Solving an equation of degree 2 or higher with no natural solutions

Many equations of degree 2 or higher can be solved just using natural numbers. For example:

$$2 + x^2 = 11$$

In order to find its solution we must to calculate $\sqrt{9}$. So the solution is $x = 3$.

But when we work with negative labels and with vectors, we know that the multiplication of two negative labels is a positive label. We know that $(-3)(-3) = 9$. So $(-3)$ is also a solution of that equation.

The underlying vector explanation is that we look for a geometrical operation, performed two times on a basis vector, which will give us a translation vector. And that translation vector will lead us from the point $(2, 0)$ to the point $(11, 0)$.

We get the following vectorial equation where X stands for an unknown geometrical operation:

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} + X \circ X(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 11 \\ 0 \end{pmatrix}$$

So the basis vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ will undergo a homothety $H_3$ and then rotation of 180°. Our solution is:

$$X = R(180°) \circ H_3$$

So the translation vector is obtained applying twice that solution to the basis vector:

$$X \circ X(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = X(\begin{pmatrix} -3 \\ 0 \end{pmatrix}) = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$$

We have sometimes equations as the following one:

$$1 + x + x^2 = 0$$

When we try to calculate the discriminant we find:

$$\Delta = 1^2 - 4 \cdot 1 \cdot 1 = -3 < 0$$

So there is no natural solution. But we can find a geometrical solution. And the question will be:

*Starting from the point (1,0), which geometrical operation X will lead to the origin O by following the steps indicated by the vectorial equation?*

We get the following vectorial equation where X stands for an unknown geometrical operation:

$$\begin{pmatrix}1\\0\end{pmatrix} + X(\begin{pmatrix}1\\0\end{pmatrix}) + X \circ X(\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}0\\0\end{pmatrix}$$

Here are the steps: the vector basis $\begin{pmatrix}1\\0\end{pmatrix}$ undergoes the operation X and we get a translation vector used to reach a second point, then the same vector basis undergoes twice the operation X and we get another translation vector, and from the second point we reach the origin 0.

We know, thanks of two centuries of studies about that kind of equations, that the solution is linked to a composition of a translation and a homothety. But what we are showing here is that the solution is not a complex number but a geometrical operation, and more precisely a composition of a homothety and a rotation. So X has the following form, where $\lambda > 0$ and $-180° \leq \theta \leq 180°$:

$$X = H_\lambda \circ R(\theta)$$

By using a matrix, we get:

$$X = \lambda \begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}$$

So the vectorial equation becomes:

$$\begin{pmatrix}1\\0\end{pmatrix} + \lambda \begin{pmatrix}cos(\theta) & -sin(\theta)\\ sin(\theta) & cos(\theta)\end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix} + \lambda^2 \begin{pmatrix}cos(\theta) & -sin(\theta)\\ sin(\theta) & cos(\theta)\end{pmatrix}^2 \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

So we get:

$$\begin{pmatrix}1\\0\end{pmatrix} + \lambda \begin{pmatrix}cos(\theta)\\ sin(\theta)\end{pmatrix} + \lambda^2 \begin{pmatrix}cos(2\theta)\\ sin(2\theta)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

To continue, we must to solve a parametric equation:

$$\begin{cases}1 + \lambda\cos(\theta) + \lambda^2\cos(2\theta) = 0\\ \lambda\sin(\theta) + \lambda^2\sin(2\theta) = 0\end{cases}$$

Knowing that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and that $\cos(2\theta) = 2\cos^2(\theta) - 1$, we get:

$$\begin{cases}1 + \lambda\cos(\theta) + \lambda^2(2\cos^2(\theta) - 1) = 0\\ \lambda\sin(\theta) + \lambda^2 2\sin(\theta)\cos(\theta) = 0\end{cases}$$

So,

$$\begin{cases}1 + \lambda\cos(\theta) + 2\lambda^2\cos^2(\theta) - \lambda^2 = 0\\ \lambda\sin(\theta) + 2\lambda^2\sin(\theta)\cos(\theta) = 0\end{cases}$$

Then, from the second equation we get:

$$\lambda\sin(\theta)(1 + 2\lambda\cos(\theta)) = 0$$

So,looking for non-trivial solutions, we solve:

$$1 + 2\lambda \cos(\theta) = 0$$

Then, working with negative labels:

$$\cos(\theta) = -\frac{1}{2\lambda}$$

We use that result in the first equation of the parametric equation and we get:

$$1 + \lambda(-\frac{1}{2\lambda}) + 2\lambda^2(-\frac{1}{2\lambda})^2 - \lambda^2 = 0$$

So,

$$1 - \frac{1}{2} + \frac{1}{2} - \lambda^2 = 0$$

So,

$$1 - \lambda^2 = 0$$

As $\lambda > 0$, the only solution is $\lambda = 1$. This means that the operation X we are looking for has no homothety. It's just a rotation. and now we will find $\theta$.

We saw above that

$$1 + 2\lambda \cos(\theta) = 0$$

So,

$$1 + 2\cos(\theta) = 0$$

Then

$$\cos(\theta) = -\frac{1}{2}$$

So, the principal solutions are $\theta_1 = 120°$ and $\theta_2 = -120°$. And finally we have 2 solutions for X:

$$X_1 = R(120°)$$

$$X_2 = R(-120°)$$

We can use also a analogue method to the completing the square. We take the vectorial equation seen above:

$$\begin{pmatrix}1\\0\end{pmatrix} + X(\begin{pmatrix}1\\0\end{pmatrix}) + X \circ X(\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}0\\0\end{pmatrix}$$

We rewrite it in the folowing way, knowing that X is a matrix:

$$\begin{pmatrix}1\\0\end{pmatrix} + X\begin{pmatrix}1\\0\end{pmatrix} + X^2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Then we rewrite it as this to have perfect square trinomial, where $I_2$ stands for the idetity matrix of size 2:

$$\frac{3}{4}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{4}I_2\begin{pmatrix}1\\0\end{pmatrix} + X\begin{pmatrix}1\\0\end{pmatrix} + X^2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Thanks to the rules of distribution of matrix, we get:

$$\frac{3}{4}\begin{pmatrix}1\\0\end{pmatrix} + (\frac{1}{2}I_2 + X)^2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

So,

$$(\frac{1}{2}I_2 + X)^2\begin{pmatrix}1\\0\end{pmatrix} = -\frac{3}{4}\begin{pmatrix}1\\0\end{pmatrix}$$

We rewrite it as:

$$(\frac{1}{2}I_2 + X)^2\begin{pmatrix}1\\0\end{pmatrix} = \frac{3}{4}\begin{pmatrix}-1\\0\end{pmatrix}$$

Let $Y = \frac{1}{2}I_2 + X$, so we get the equation:

$$Y^2\begin{pmatrix}1\\0\end{pmatrix} = \frac{3}{4}\begin{pmatrix}-1\\0\end{pmatrix}$$

Thanks to what we have seen before, we know that Y must be a composition of a rotation and a homothety, such that if it is applied twice to the vector basis $\begin{pmatrix}1\\0\end{pmatrix}$, we get $\frac{3}{4}\begin{pmatrix}-1\\0\end{pmatrix}$. So this time we need a rotation of 90° composed with a homothety $H_{\frac{\sqrt{3}}{2}}$, and also a rotation of $-90°$ with a homothety $H_{\frac{\sqrt{3}}{2}}$. So $Y_1 = H_{\frac{\sqrt{3}}{2}} \circ R(90°)$.

So,

$$(\frac{1}{2}I_2 + X_1)\begin{pmatrix}1\\0\end{pmatrix} = H_{\frac{\sqrt{3}}{2}} \circ R(90°)\begin{pmatrix}1\\0\end{pmatrix}$$

And then,

$$X_1\begin{pmatrix}1\\0\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} + H_{\frac{\sqrt{3}}{2}} \circ R(90°)\begin{pmatrix}1\\0\end{pmatrix}$$

After the rotation of 90° of the vector basis $\begin{pmatrix}1\\0\end{pmatrix}$, we can rewrite it as:

$$X_1\begin{pmatrix}1\\0\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} + \frac{\sqrt{3}}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

Now we can describe the operation $X_1$: it is the operation such that if it is aplied to the vector basis $\begin{pmatrix}1\\0\end{pmatrix}$, we get the vector $\begin{pmatrix}-\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix}$.

As we know that $X_1 = H_\lambda \circ R(\theta_1)$, we will have:

$$\lambda = \sqrt{(-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$\theta_1 = \arctan(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}})$$

So we will have:

$$\lambda = 1$$
$$\theta_1 = 120^\circ)$$

With the solution $X_1$ we can represent the path from the the point $A(1, 0)$ to the origin O. It can be also represented as a path from to origin to the origin, through the points A and B. This path is constructed with the addition of the vectors we get in each step of the vector equation. See the figure 2

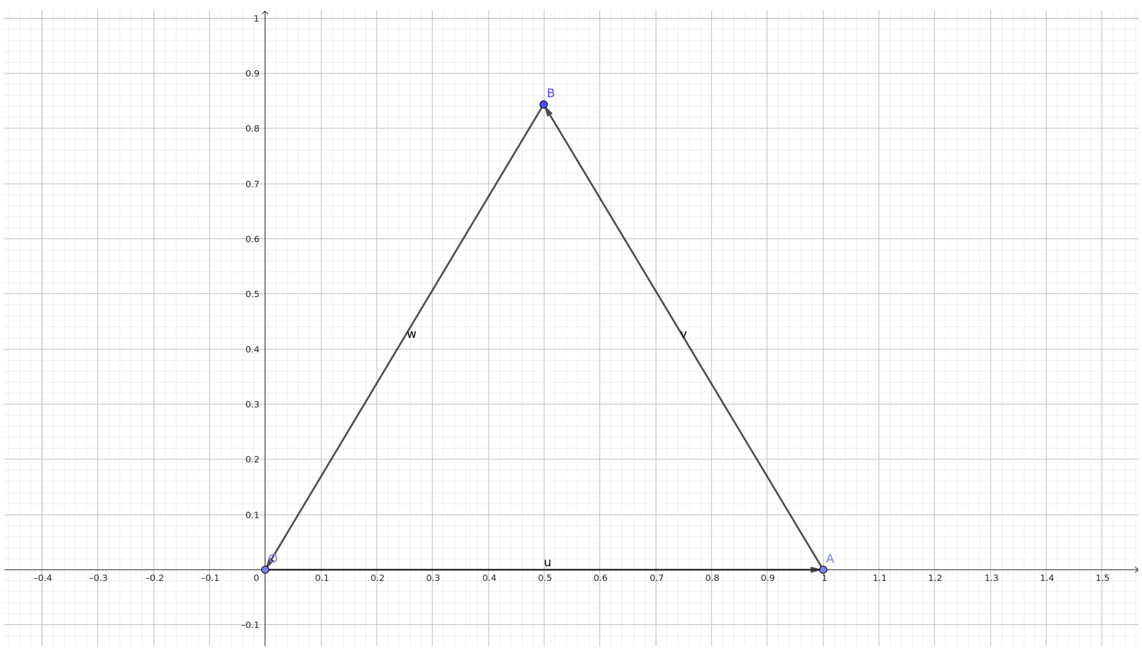


Figure 2: Representation of the path given by a solution to the vector equation

As we have seen before, there will be another solution $X_2$ with:

$$\lambda = 1$$
$$\theta_1 = -120^\circ$$

As we have seen, negative numbers have a geometrical underlying structure. That is why geometry appears again when we try to solve some polynomial equations.

# 7 The vector nature of the formulas of Euler and De Moivre

## 7.1 The formula of De Moivre

As we have seen that, if we apply an operation X, which is a rotation, to the vector basis $\begin{pmatrix}1\\0\end{pmatrix}$, we get:

$$X(\begin{pmatrix}1\\0\end{pmatrix}) = R(\theta)(\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}\cos(\theta)\\\sin(\theta)\end{pmatrix}$$

We can rewrite it as this:

$$X(\begin{pmatrix}1\\0\end{pmatrix}) = R(\theta)(\begin{pmatrix}1\\0\end{pmatrix}) = \cos(\theta)\begin{pmatrix}1\\0\end{pmatrix} + \sin(\theta)\begin{pmatrix}0\\1\end{pmatrix}$$

Now, if we apply n times the operation X to that vector basis, we get:

$$X^n(\begin{pmatrix}1\\0\end{pmatrix}) = R^n(\theta)(\begin{pmatrix}1\\0\end{pmatrix}) = \cos(n\theta)\begin{pmatrix}1\\0\end{pmatrix} + \sin(n\theta)\begin{pmatrix}0\\1\end{pmatrix}$$

And that corresponds to what De Moivre found with his formula:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

In fact, the underlying nature of the complex number i is linked the vector basis $\begin{pmatrix}0\\1\end{pmatrix}$. The fact of multiplying $\cos(\theta) + i\sin(\theta)$ to itself n times produces n-1 rotations, and so a multiplication of $\theta$ by n.

## 7.2 The formulas of Euler

We know that:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ... + \frac{1}{k!}x^k + ...$$

This formula can be seen as a schema for a vector version. We will work with another function called $Exp_v$. If it is applied to the result of an operation X, we get:

$$Exp_v(X(\begin{pmatrix}1\\0\end{pmatrix})) = \begin{pmatrix}1\\0\end{pmatrix} + X(\begin{pmatrix}1\\0\end{pmatrix}) + \frac{1}{2!}X^2(\begin{pmatrix}1\\0\end{pmatrix}) + \frac{1}{3!}X^3(\begin{pmatrix}1\\0\end{pmatrix}) + ...$$

We will apply that function to $X = H_\lambda \circ R(90°)(\begin{pmatrix}1\\0\end{pmatrix})$:

$$
\begin{aligned}
Exp_v(H_\lambda \circ R(90°)(\begin{pmatrix}1\\0\end{pmatrix})) &= \begin{pmatrix}1\\0\end{pmatrix} + H_\lambda \circ R(90°)(\begin{pmatrix}1\\0\end{pmatrix}) + \frac{1}{2!}H_\lambda^2 \circ R^2(90°)(\begin{pmatrix}1\\0\end{pmatrix}) + \frac{1}{3!}H_\lambda^3 \circ R^3(90°)(\begin{pmatrix}1\\0\end{pmatrix}) + ...\\
&= \begin{pmatrix}1\\0\end{pmatrix} + H_\lambda(\begin{pmatrix}0\\1\end{pmatrix}) + \frac{1}{2!}H_\lambda^2(\begin{pmatrix}-1\\0\end{pmatrix}) + \frac{1}{3!}H_\lambda^3(\begin{pmatrix}0\\-1\end{pmatrix}) + \frac{1}{4!}H_\lambda^4(\begin{pmatrix}1\\0\end{pmatrix}) + \frac{1}{5!}H_\lambda^5(\begin{pmatrix}0\\1\end{pmatrix}) + ...\\
&= \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\\lambda\end{pmatrix} + \frac{1}{2!}\begin{pmatrix}-\lambda^2\\0\end{pmatrix} + \frac{1}{3!}\begin{pmatrix}0\\-\lambda^3\end{pmatrix}) + \frac{1}{4!}\begin{pmatrix}\lambda^4\\0\end{pmatrix}) + \frac{1}{5!}\begin{pmatrix}0\\\lambda^5\end{pmatrix} + \frac{1}{6!}\begin{pmatrix}-\lambda^6\\0\end{pmatrix} + ...\\
&= \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\\lambda\end{pmatrix} + \begin{pmatrix}-\frac{1}{2!}\lambda^2\\0\end{pmatrix} + \begin{pmatrix}0\\-\frac{1}{3!}\lambda^3\end{pmatrix}) + \begin{pmatrix}\frac{1}{4!}\lambda^4\\0\end{pmatrix}) + \begin{pmatrix}0\\\frac{1}{5!}\lambda^5\end{pmatrix} + \begin{pmatrix}-\frac{1}{6!}\lambda^6\\0\end{pmatrix} + ...\\
&= \begin{pmatrix}1 - \frac{1}{2!}\lambda^2 + \frac{1}{4!}\lambda^4 - \frac{1}{6!}\lambda^6 + ...\\ \lambda - \frac{1}{3!}\lambda^3 + \frac{1}{5!}\lambda^5 - ...\end{pmatrix}\\
&= \begin{pmatrix}\cos(\lambda)\\\sin(\lambda)\end{pmatrix}\\
&= \cos(\lambda)\begin{pmatrix}1\\0\end{pmatrix} + \sin(\lambda)\begin{pmatrix}0\\1\end{pmatrix}
\end{aligned}
$$

So we have found the vector expression of what Euler found using $ix$:

$$e^{ix} = \cos(x) + i\sin(x)$$

As we saw before, the complex number i is linked the vector basis $\begin{pmatrix}0\\1\end{pmatrix}$.
And finaly, if $\lambda = \pi$, we get with the vector function :

$$
\begin{aligned}
Exp_v(H_\pi \circ R(90°)(\begin{pmatrix}1\\0\end{pmatrix})) &= \cos(\pi)\begin{pmatrix}1\\0\end{pmatrix} + \sin(\pi)\begin{pmatrix}0\\1\end{pmatrix}\\
&= -1\begin{pmatrix}1\\0\end{pmatrix}
\end{aligned}
$$

And that corresponds to what Euler found with his formula:

$$e^{i\pi} = -1$$

## 8 The meaning of the other number line

As we have seen, at the beginning negative labels were created to perform calculations with gains and debts. We have seen that those calculation are intrinsically geometric and we have found a second axis which is also a number line. That number line can be used to describe the gains and losses of a second agent. For exemple, John has \$100 and his spends little by little his money by buying objects in a store. When John losses money, the store gets the money. And there will be a

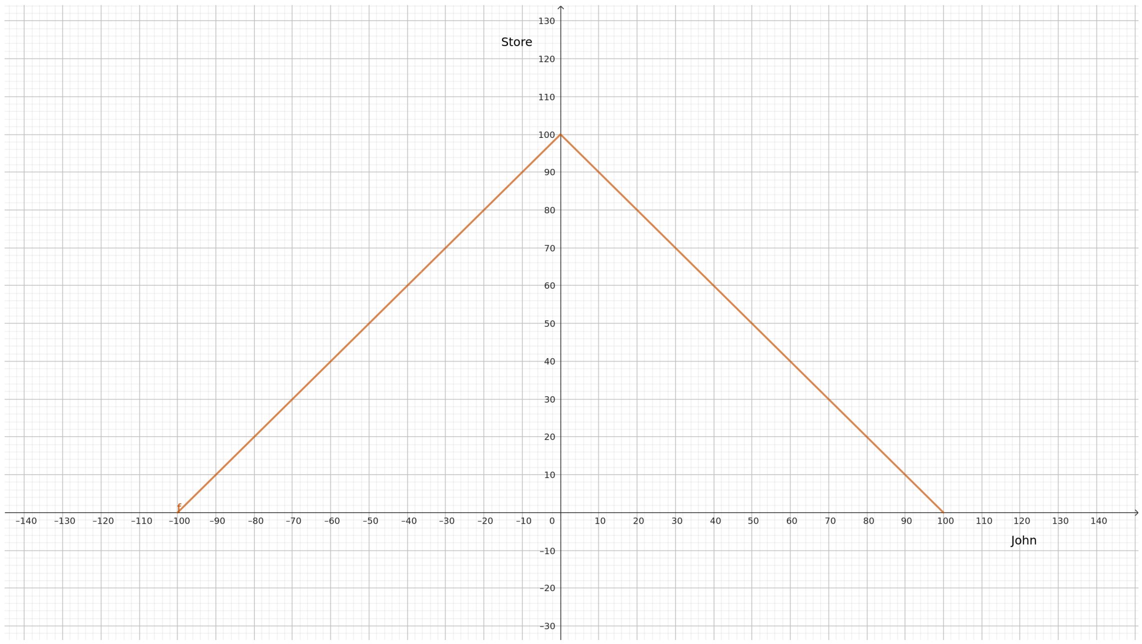


Figure 3: Representation of the gains and losses of 2 agents

moment in which John has no more money, and the store could give him some credit. But in doing that, the store losses money. That situation can be represented as in figure 3:

The other axis can be used to represent the situation in pole vaulting. The pole vaulter has a gain of height as he losses some meters from the standards above which there is the bar. When he is on the other side of the standards, he has a negative position from the standards and he losses height. That situation can be represented as in figure 4 (in meters):

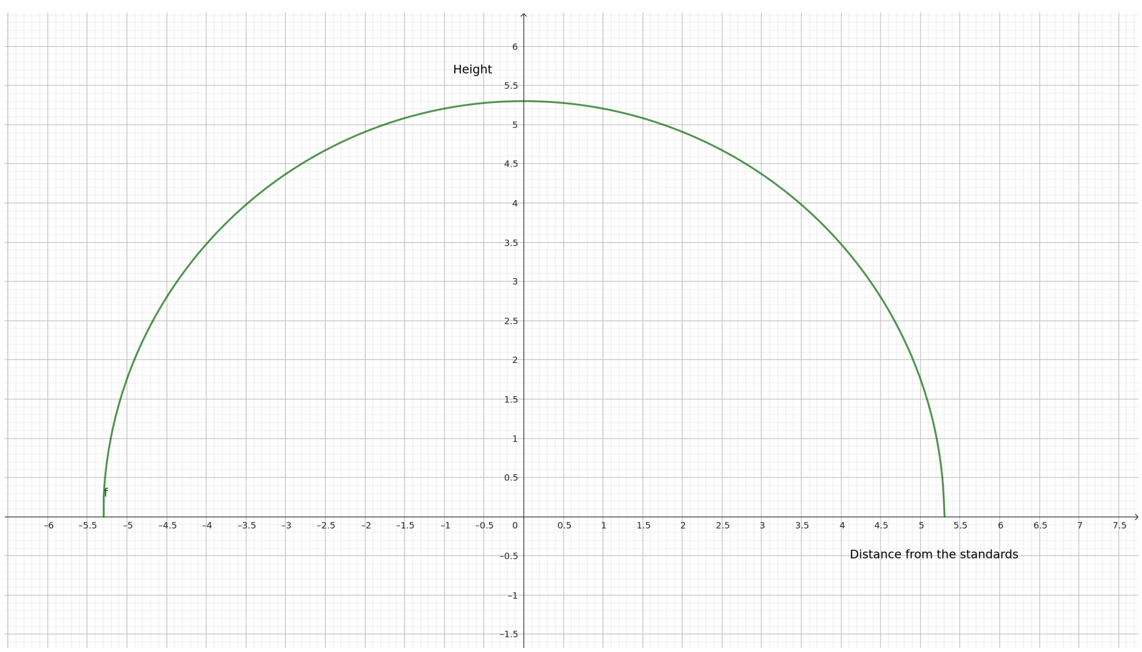


Figure 4: Representation of the gains and losses of distance and height

## 9 Conclusion

So we have seen that there are no negative numbers nor complex numbers. There are actually geometrical operations which are the solutions of certain arithmetical and polynomial problems.

## References


[Brahmagupta] Brahmagupta, Bhaskaracarya, b. 1114; Colebrooke, H. T. (Henry Thomas), *Algebra, with Arithmetic and mensuration, from the Sanscrit of Brahmegupta and Bháscara.* , John Murray, Albemarle Street, London, 1817.

[Cajori1991] Cajori, F., *A history of mathematics*, AMS Chelsea Publishing, 5th edition, New York,1991.

[Chemla] Chemla, K. C., *East Asian mathematics*, Encyclopedia Britannica, https://www.britannica.com/science/East-Asian-mathematics/The-great-early-period-1st-7th-centuries, page consulted on August 2025.

[Kangshen2000] Kangshen, S. (Editor), Crossley, J. N. (Translator), Lun, A. W. -C. (Translator) , *The Nine Chapters on the Mathematical Art: Companion and Commentary*, 1st Edition, Oxford University Press, 2000

[Katz2009] Katz, V. J., *A history of mathematics, an introduction*, (3rd ed.), Addison-Wesley,2009.

[LakoffNúñez2000] Lakoff, G., Núñez, R., *Where Mathematics come from*, Basic Books,2000.

[Martzloff2006] Martzloff, J.-C., *A History of Chinese Mathematics*, Springer-Verlag Berlin Heidelberg, 2006

[Mattessich1998] Mattessich, R., *"From accounting to negative numbers: A signal contribution of medieval India to mathematics,"*, Accounting Historians Journal: Vol. 25 : Iss. 2 , Article 9, 1998.

[Mumford] Mumford, D., *What's so Baffling About Negative Numbers? – a Cross-Cultural Comparison*, https://www.dam.brown.edu/people/mumford/beyond/papers/2010b–Negatives-PrfShts.pdf, page consulted on August 2025.

[Smith2001] Smith, Martha K., *History of Negative Numbers*, https://web.ma.utexas.edu/users/mks/326K/Negnos.html, 2001, page consulted on August 2025.