The 3D fractal superset which contains the Mandelbrot set without complex numbers

Jaime Vladimir Torres-Heredia Julca*
torres.heredia@concept-global.net

December 31, 2025

Abstract

In this paper we will see that each vector of the 3D Euclidean vector space can be expressed with operations involving rotations of the unit vector of the x-axis. Thanks to that, we will define a new multiplication between vectors which is analogue to what we have seen in our previous paper viXra:2510.0152 without complex numbers. This operation will allow us to construct a 3D fractal set which contains the Mandelbrot set in the planes OXY and OXZ. We will show some cross sections of other parts of that 3D fractal set.

1 Introduction

Some fractals, involving complex numbers, were discovered by mathematicians as Gaston Julia and Benoît Mandelbrot (see [WikipediaM]). In ViXra:2508.0176 we saw that we don't need the concept of complex number nor the concept of negative number, and we worked with vectors and geometrical operations. In viXra:2510.0152 we saw that we can define vector operations which are analogue to the operations involving complex numbers and those new operations allowed us to construct the Mandelbrot set in the Euclidean vector space. In this paper we will see that we can construct a new vector multiplication in the 3D Euclidean vector space which will allow us to construct a fractal 3D set which contains the Mandelbrot set.

2 Expressing a 3D vector with operations involving rotations of the unit vector of the x-axis

In this framework we will work with the 3D Euclidean vector space and with the x-axis, the y-axis and the z-axis (see the figure 1).

A vector in the 3D Euclidean vector space can be expressed as this, in a way analogous to what we have seen in our previous paper (see [Torres-Heredia2025O]), where $R_z(90°)$ is a rotation matrix about the z-axis and $R_y(90°)$ is a rotation matrix about the y-axis :

*Jaime V. Torres-Heredia is an independent researcher. He studied philosophy and computer science for the humanities at the University of Geneva, and holds a M.A. He is also a self-taught mathematician.

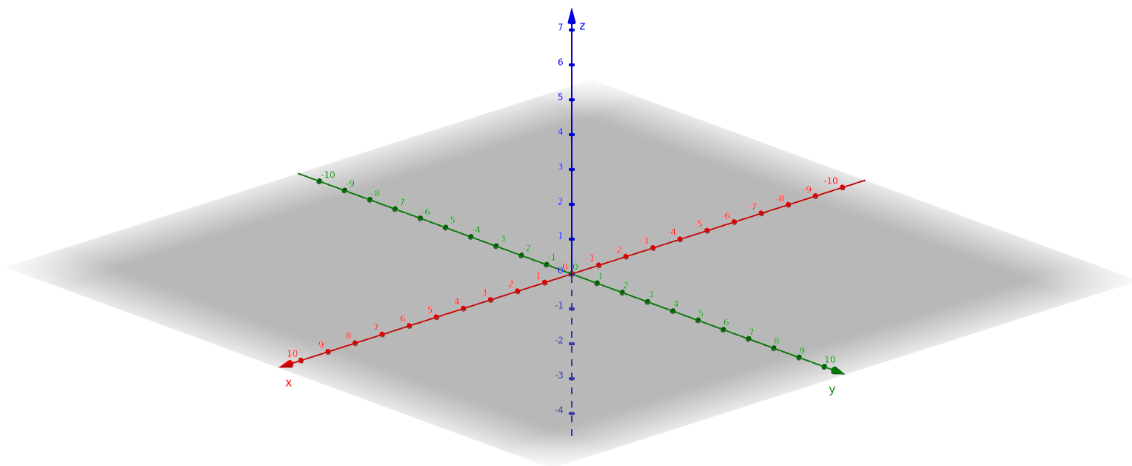

Figure 1: The axis of the 3D Euclidean vector space

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + bR_z(90°) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + cR_y(90°) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The direction of vector rotation is counterclockwise if the angle θ is positive. And it is important to note that in the 3D space the composition of rotations is not commutative.

3 The operations with the mathematical objects and the resulted vectors of these operations

3.1 The addition

In this vector space, we can perform additions of vectors with the usual rule:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$$

3.2 The new multiplication

As we have seen before, each vector of the 3D space can be expressed as this:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + bR_z(90^\circ)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + cR_y(90^\circ)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

If we have a second vector, it can be expressed in the same way:

$$\begin{pmatrix} d \\ e \\ f \end{pmatrix} = d\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + eR_z(90^\circ)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + fR_y(90^\circ)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Then we rewrite the 2 vectors as this, where I_3 stands for the identity matrix of size 3:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = aI_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + bR_z(90^\circ)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + cR_y(90^\circ)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} d \\ e \\ f \end{pmatrix} = dI_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + eR_z(90^\circ)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + fR_y(90^\circ)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Now, thanks to the rules of distribution of matrices, we get:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (aI_3 + bR_z(90^\circ) + cR_y(90^\circ))\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} d \\ e \\ f \end{pmatrix} = (dI_3 + eR_z(90^\circ) + fR_y(90^\circ))\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Now we can compose those two operations thanks to matrix multiplications, using the rules of matrix distribution:

$$(aI_3+bR_z(90^\circ)+cR_y(90^\circ))(dI_3+eR_z(90^\circ)+fR_y(90^\circ)) = aI_3dI_3+aI_3eR_z(90^\circ)+aI_3fR_y(90^\circ)+bR_z(90^\circ)dI_3$$
$$+ bR_z(90^\circ)eR_z(90^\circ) + bR_z(90^\circ)fR_y(90^\circ) + cR_y(90^\circ)dI_3 + cR_y(90^\circ)eR_z(90^\circ) + cR_y(90^\circ)fR_y(90^\circ)$$

So,

$$(aI_3+bR_z(90^\circ)+cR_y(90^\circ))(dI_3+eR_z(90^\circ)+fR_y(90^\circ)) = adI_3+aeR_z(90^\circ)+afR_y(90^\circ)+bdR_z(90^\circ)$$
$$+ beR_z(90^\circ)R_z(90^\circ) + bfR_z(90^\circ)R_y(90^\circ) + cdR_y(90^\circ) + ceR_y(90^\circ)R_z(90^\circ) + cfR_y(90^\circ)R_y(90^\circ)$$

Then,

$$(aI_3+bR_z(90°)+cR_y(90°))(dI_3+eR_z(90°)+fR_y(90°)) = adI_3+aeR_z(90°)+afR_y(90°)+bdR_z(90°) + beR_z(180°) + bfR_z(90°)R_y(90°) + cdR_y(90°) + ceR_y(90°)R_z(90°) + cfR_y(180°)$$

So,

$$(aI_3+bR_z(90°)+cR_y(90°))(dI_3+eR_z(90°)+fR_y(90°)) = adI_3+(ae+bd)R_z(90°)+(af+cd)R_y(90°) + beR_z(180°) + bfR_z(90°)R_y(90°) + ceR_y(90°)R_z(90°) + cfR_y(180°)$$

And now we can apply this operation to unit vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$:

$$(adI_3 + (ae+bd)R_z(90°) + (af+cd)R_y(90°) + beR_z(180°) + bfR_z(90°)R_y(90°) + ceR_y(90°)R_z(90°)+cfR_y(180°))\begin{pmatrix}1\\0\\0\end{pmatrix} = adI_3\begin{pmatrix}1\\0\\0\end{pmatrix}+(ae+bd)R_z(90°)\begin{pmatrix}1\\0\\0\end{pmatrix}+(af+cd)R_y(90°)\begin{pmatrix}1\\0\\0\end{pmatrix} + beR_z(180°)\begin{pmatrix}1\\0\\0\end{pmatrix} + bfR_z(90°)R_y(90°)\begin{pmatrix}1\\0\\0\end{pmatrix} + ceR_y(90°)R_z(90°)\begin{pmatrix}1\\0\\0\end{pmatrix} + cfR_y(180°)\begin{pmatrix}1\\0\\0\end{pmatrix}$$

Then we get:

$$ad\begin{pmatrix}1\\0\\0\end{pmatrix} + (ae+bd)\begin{pmatrix}0\\1\\0\end{pmatrix} + (af+cd)\begin{pmatrix}0\\0\\1\end{pmatrix} +be\begin{pmatrix}-1\\0\\0\end{pmatrix}+bfR_z(90°)\begin{pmatrix}0\\0\\1\end{pmatrix}+ceR_y(90°)\begin{pmatrix}0\\1\\0\end{pmatrix}+cf\begin{pmatrix}-1\\0\\0\end{pmatrix} = ad\begin{pmatrix}1\\0\\0\end{pmatrix}+(ae+bd)\begin{pmatrix}0\\1\\0\end{pmatrix}+(af+cd)\begin{pmatrix}0\\0\\1\end{pmatrix} + be\begin{pmatrix}-1\\0\\0\end{pmatrix} + bf\begin{pmatrix}0\\0\\1\end{pmatrix} + ce\begin{pmatrix}0\\1\\0\end{pmatrix} + cf\begin{pmatrix}-1\\0\\0\end{pmatrix}$$

So, finally we get:

$$\begin{pmatrix}ad-be-cf\\ae+bd+ce\\af+cd+bf\end{pmatrix}$$

As we have seen, we have performed multiplications of matrix to get our result. For practical reasons, we can define a new multiplication between vectors, called $*$ in this framework:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} * \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} ad - be - cf \\ ae + bd + ce \\ af + cd + bf \end{pmatrix}$$

3.3 The exponentiation

We can define a new exponentiation if we multiply a vector by himself several times:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}^n = \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \end{pmatrix} * \cdots * \begin{pmatrix} a \\ b \\ c \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{n\ times}$$

4 Polynomials

The mathematical objects we were studying in our previous papers come from polynomial vector equations (see [Torres-Heredia2025A]). With the vectors and the operations defined here, we can construct also structures analogous to those polynomials defined in $\mathbb{E}^3$. So we can have a structure as:

$$P(\vec{x}) = \vec{a_0} + \vec{a_1} * \vec{x} + \vec{a_2} * \vec{x}^2 + \vec{a_3} * \vec{x}^3 + \vec{a_4} * \vec{x}^4$$

So $\vec{P}$ is also a vector.

5 The 3D fractal set which contains the Mandelbrot set

Now we can construct the 3D fractal set which contains the Mandelbrot set only with vectors and the new operations of this framework in $\mathbb{E}^3$. We will continue by following an analogous way to what has been done with complex numbers (see [WikipediaM] and [Torres-Heredia2025O]).

Let $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ and $\vec{x_0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

We will perform an iteration of a quadratic map to get the vectors $\vec{x_k}$:

$$\vec{x_k} = \vec{x_{k-1}}^2 + \vec{c}$$

And we will see if the vector norm of $\vec{x_k}$ remains bounded for all $k > 0$. If it is the case, we will add the vector $\vec{c}$ to the 3D fractal set.

So, we must to apply that iteration to several values of $\vec{c}$ and we will have the 3D fractal set in $\mathbb{E}^3$.

We will show now more details about those calculations. Firstly, the square of a vector will give a simpler expression:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}^2 = \begin{pmatrix} x^2 - y^2 - z^2 \\ 2xy + yz \\ 2xz + yz \end{pmatrix}$$

Now, we are going to represent grapically cross sections of the fractal 3D set, which are planes. For that, we will adapt the "escape time" algorithm used for the Mandelbrot set (see [WikipediaP]).

So we must to fix the value of one of the 3 components of the vector $\vec{c}$ in order to have the representation of a plane. For example, we can fix the component c_3 with a value z_0 in order to work with a xy plane.

After that, we test different values of the vector $\vec{c}$ with the iterations. If the 3D norm of $\vec{x_k}$ remains bounded for all $k > 0$, then we add $\vec{c}$ to the plane subset which is a part of the 3D fractal set. And we can represent graphically this subset in the plane xy with $z = z_0$. Obviously, we represent only the terminal points of each vector $\vec{c}$.

Now, first of all, we get the Mandelbrot set in the plane OXY, when $Z = 0$, (see the figure 2)

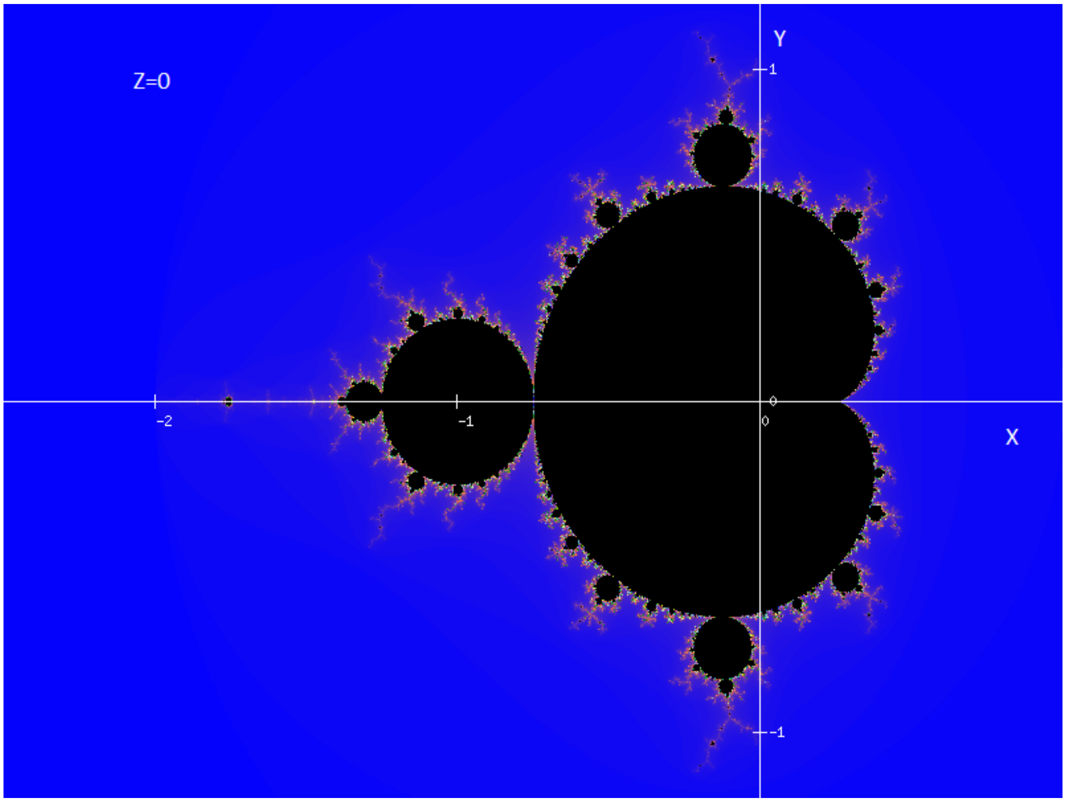

Figure 2: The Mandelbrot set in the plane OXY, when Z=0

We will see now some cross sections for different values of Z. If $Z = 0.1$, we get a form quite similar to the Mandelbrot set (see the figure 3).

For $Z = 0.3$, we get another form (see the figure 4).

For $Z = 0.5$, we get another form (see the figure 5).

We get also the Mandelbrot set in the plane OXZ, when $Y = 0$, (see the figure 6)

If $Y = 0.3$, we get a form quite similar to the Mandelbrot set (see the figure 7).

We will see now some cross sections for different values of X in the plane YZ. If $X = 0$, we get another form (see the figure 8).

If $X = -1$, we get another form (see the figure 9).

This last form correspond to the kind of disc of the Mandelbrot set centered at around $X = -1$. This form is perpendicular the one of the Mandelbrot set. Its form confirms the consistency of the cross sections presented here.

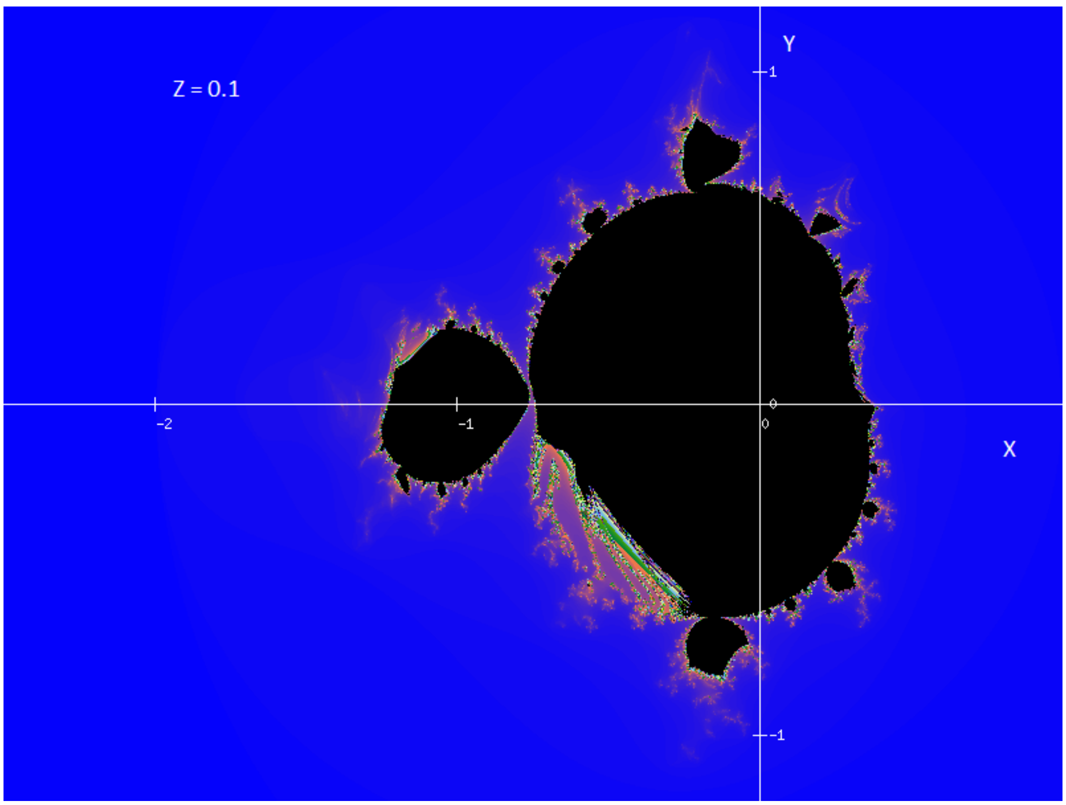

Figure 3: A subset in the plane XY, when Z=0.1

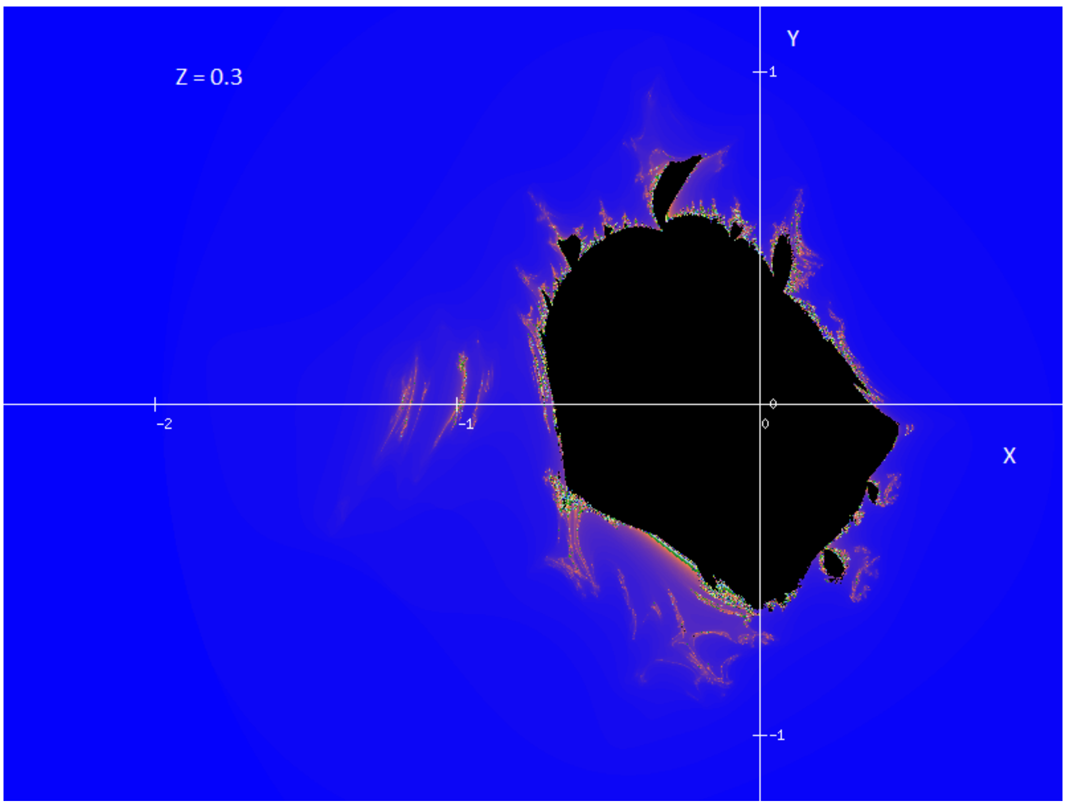

Figure 4: A subset in the plane XY, when Z=0.3

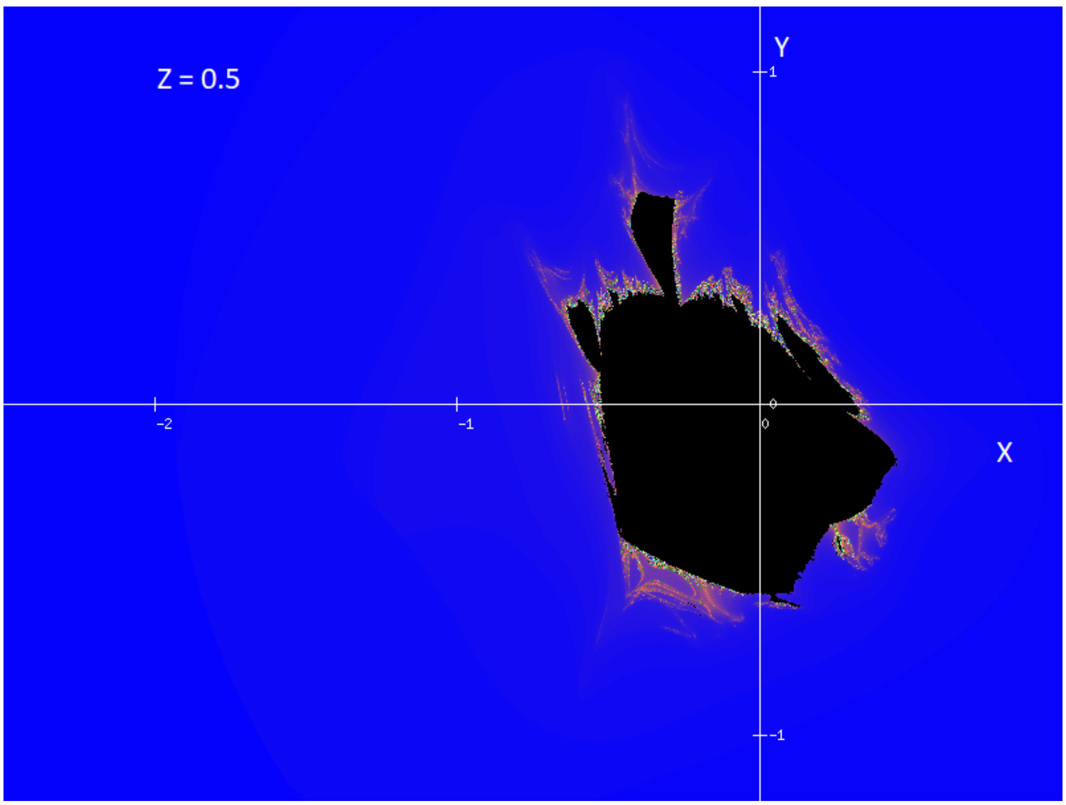

Figure 5: A subset in the plane XY, when Z=0.5

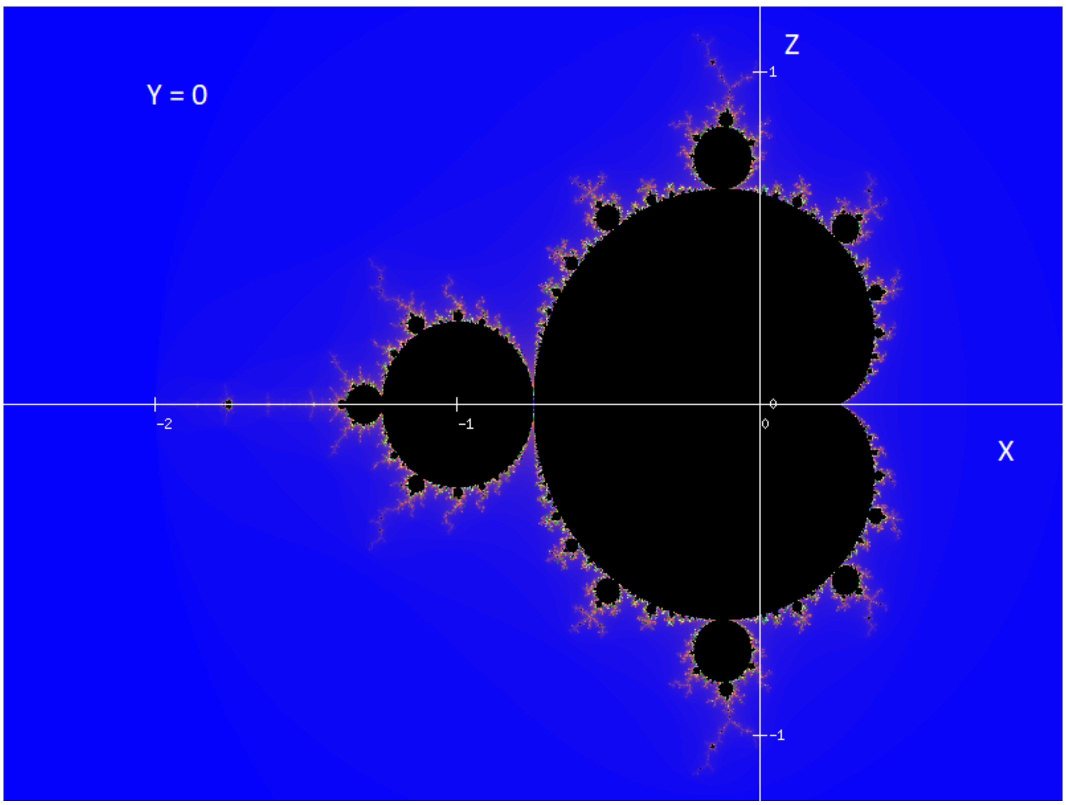

Figure 6: The Mandelbrot set in the plane OXZ, when Y=0

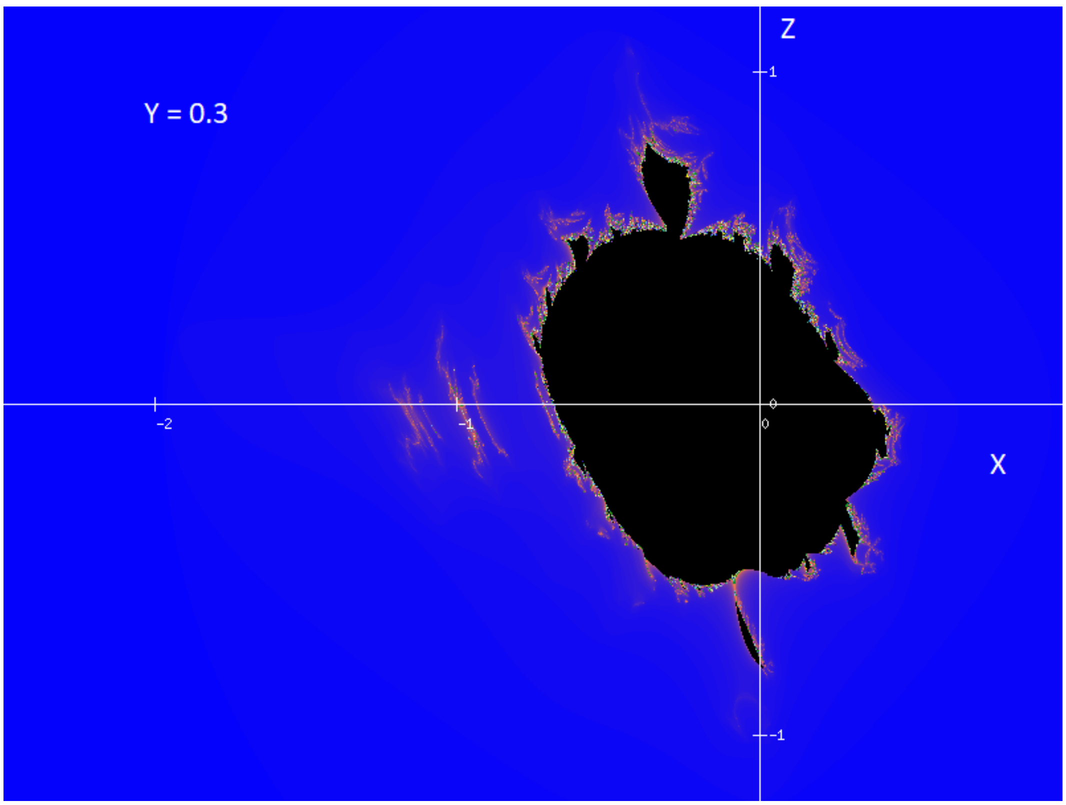

Figure 7: A subset in the plane XZ, when Y=0.3

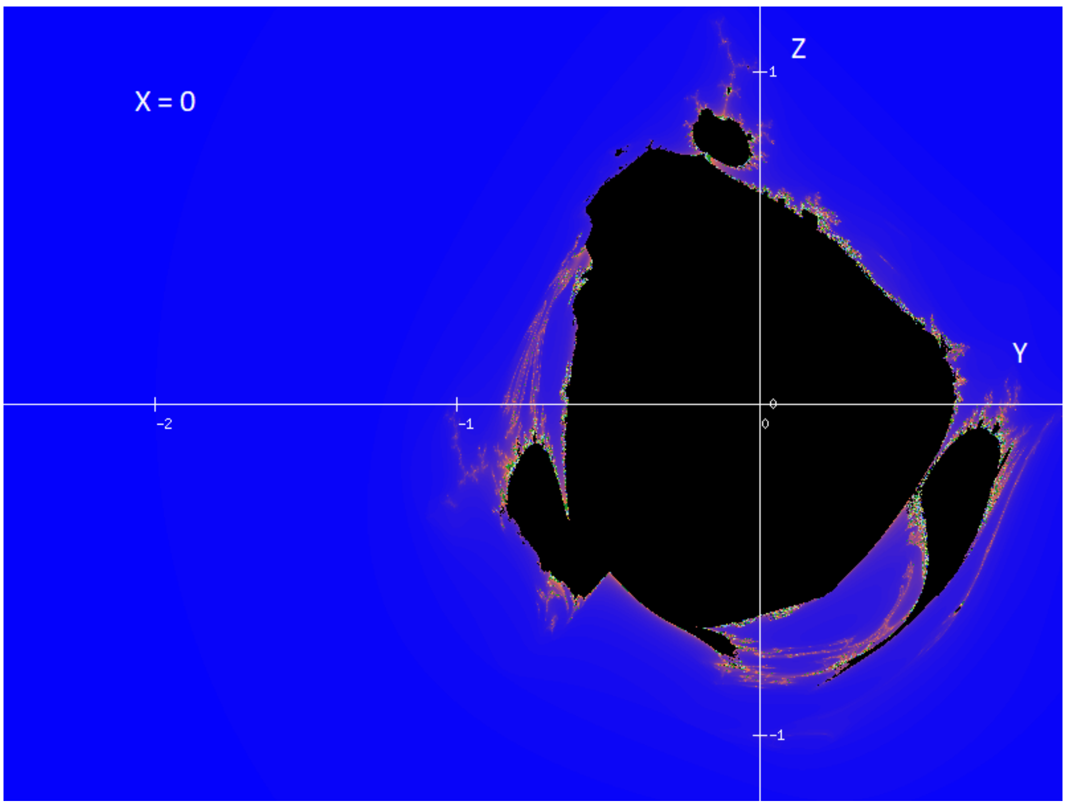

Figure 8: A subset in the plane YZ, when X=0

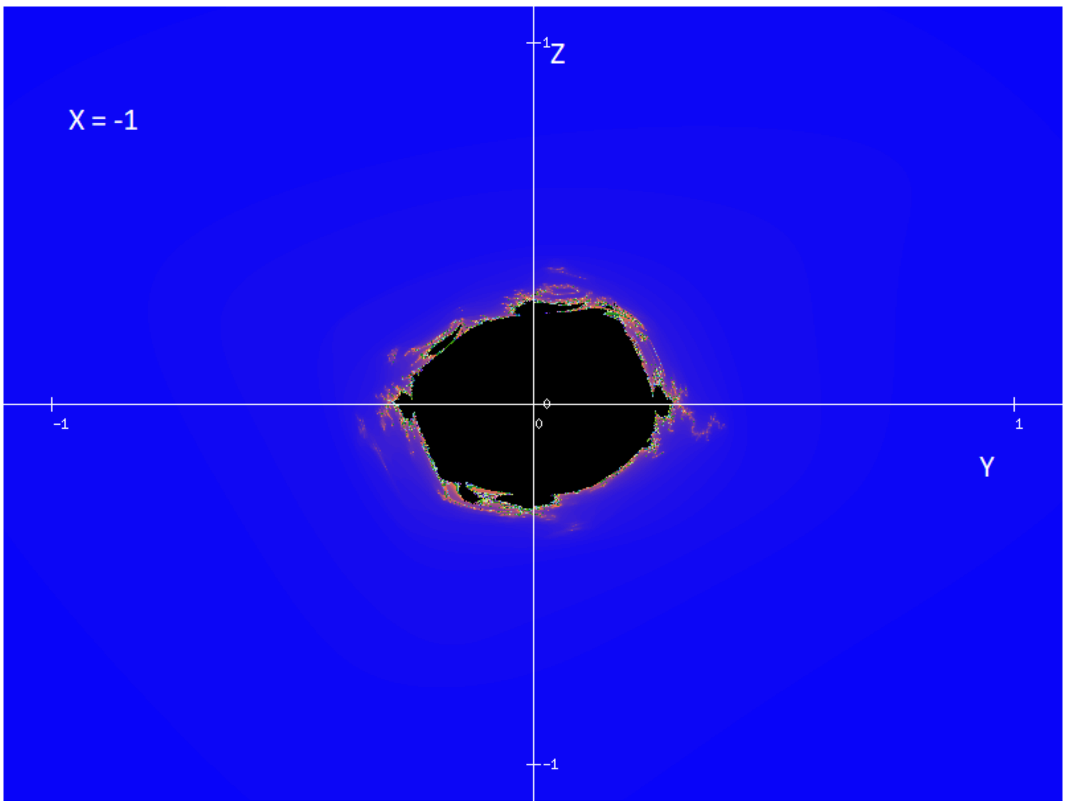

Figure 9: A subset in the plane YZ, when X=-1

6 Conclusion

So we have seen that there is a 3D superset containing the Mandelbrot set in the Euclidean vector space.

References

[Ahlfors1979] Ahlfors, L., *Complex analysis. An introduction to the theory of analytic functions of one complex variable.*, third edition, McGraw-Hill, New York, 1979.

[Comm02] Comissions romandes de mathématiques, de physique et de chimie *Formulaires et tables - Mathématiques Physique Chimie*, Editions du Tricorne, 2002.

[Torres-Heredia2025A] Torres-Heredia Julca, J., *Avoiding Negative Numbers and Complex Numbers Thanks to the Study of the Geometrical Nature of Some Arithmetical and Polynomial Problems*, viXra:2508.0176, 2025, https://vixra.org/abs/2508.0176

[Torres-Heredia2025O] Torres-Heredia Julca, J., *Operations, Analysis and Fractals Without Complex Numbers*, viXra:2510.0152, 2025, https://vixra.org/abs/2510.0152

[WikipediaC] Wikipedia, *Complex number*, page consulted on December 2025.

[WikipediaM] Wikipedia, *Mandelbrot set*, page consulted on December 2025.

[WikipediaP] Wikipedia, *Plotting algorithms for the Mandelbrot set*, page consulted on December 2025.