On the intrinsic geometrical nature of the Fundamental Theorem of Algebra without complex numbers

Jaime Vladimir Torres-Heredia Julca*
torres.heredia@concept-global.net

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Abstract

In this paper we are going to study the polynomials whose arguments and coefficients are vectors in the Euclidean vector space together with the new operations defined in our previous paper viXra:2510.0152. In order to prove the Fundamental Theorem of Algebra with topological tools, we are going to define the limits and derivatives with respect to vectors. We are going to represent some values of the polynomials thanks to paths in the plane. We will see that for the partial sums of the Taylor development linked to the exponential function, we get spiral paths leading to the unit circle. We are going to find the zeros/roots and we are going to present a new formulation of the Fundamental Theorem of Algebra, in the Euclidean vector space, with its meaning linked to paths in the plane. We are going to adapt a proof made by Laurent Schwartz with complex numbers. We are going to present also an adaptation of the algorithm of Kneser in order to find the roots. We will show that the Fundamental Theorem of Algebra is definitely geometrical. We will give also a link to a code source for GNU Octave for experiments with operations and polynomials in this framework.

1 Introduction

According to its formulation linked to complex numbers, the Fundamental Theorem of Algebra "states that every non-constant single-variable polynomial with complex coefficients has at least one complex root" (see [WikipediaF]). This theorem has been proved during the last two centuries following several ways (see [RahmanSchmeisser2002]).

Many mathematicians have noted that the proofs need geometrical and even topological considerations (see [WikipediaF]) and (see [RahmanSchmeisser2002], p. 62). So it is not really a theorem of Algebra.

And in fact, we will show that it is definitely not a theorem of Algebra because, as we have seen in our previous papers, the polynomials studied are actually geometrical. We will prove that theorem with vectors and topological tools.

2 The zeros or the roots of the polynomials

The mathematical objects we are studying come from polynomial equations, as we have seen in our first paper (see [Torres-Heredia2025A]). So we can have a polynomial of degree 4 as:

*Jaime V. Torres-Heredia is an independent researcher. He studied philosophy and computer science for the humanities at the University of Geneva, and holds a M.A. He is also a self-taught mathematician.

$$P(\vec{x}) = \vec{a_0} + \vec{a_1} * \vec{x} + \vec{a_2} * \vec{x}^2 + \vec{a_3} * \vec{x}^3 + \vec{a_4} * \vec{x}^4$$

So P is also a vector and we can write also $\vec{P}$. And we can find the zeros of this polynomial with several methods analogous to those which were used for complex numbers. As we have seen in our previous paper (see [Torres-Heredia2025O]), $\mathbb{E}^2$, together with the 2 operations $+$ and $*$ defined in this framework, is a field. So the rules and theorems (the Division Algorithm, the Remainder Theorem, the Factor Theorem, etc.) for polynomials over a field apply here (see [Ayres1965], pp. 127-128).

For example, let $P = \vec{x}^2 + 2\vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We can check that the vector $-\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a zero of the polynomial:

$$\begin{aligned} P(-\begin{pmatrix} 1 \\ 0 \end{pmatrix}) &= P(\begin{pmatrix} -1 \\ 0 \end{pmatrix}) \\ &= \begin{pmatrix} -1 \\ 0 \end{pmatrix}^2 + 2\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So the polynomial $P(\vec{x})$ can be divided by $\vec{x} - (-\begin{pmatrix} 1 \\ 0 \end{pmatrix})$:

$$\begin{array}{rr|l} - & \vec{x}^2 + 2\vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \cline{3-3} & \vec{x}^2 + \vec{x} \phantom{+ \begin{pmatrix} 1 \\ 0 \end{pmatrix}} & \vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \cline{2-2} - & \vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \\ & \vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \\ \cline{2-2} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \end{array}$$

And so $P(\vec{x}) = (\vec{x} - (-\begin{pmatrix} 1 \\ 0 \end{pmatrix})) * (\vec{x} - (-\begin{pmatrix} 1 \\ 0 \end{pmatrix})) = (\vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}) * (\vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = (\vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix})^2$. And, in fact, $-\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a zero of $P(\vec{x})$ of the order 2.

3 The norm and some inequalities

As we are working in $\mathbb{E}^2$, we will use the euclidean norm defined in it. We will have, so:

$$\parallel \begin{pmatrix} a \\ b \end{pmatrix} \parallel = \sqrt{a^2 + b^2}$$

We wil have also the triangle inequality:

$$\| \vec{x} + \vec{y} \| \leqslant \| \vec{x} \| + \| \vec{y} \|$$

And the folowing equality, where λ is a number:

$$\| \lambda \vec{x} \| = | \lambda | \| \vec{x} \|$$

Now, we are going to show that $\| \vec{x} * \vec{y} \| = \| \vec{x} \| \| \vec{y} \|$ with the operation $*$ defined in our previous paper (see [Torres-Heredia2025O]). Firstly, we know that we consider each vector as the result of the composition of a homothety and a rotation of the unit vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. So we can rewrite, using matrices:

$$\vec{x} = \lambda R(\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\vec{y} = \mu R(\varphi) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We know that the result of the rotation of the unit vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ will be a vector with the same norm. And this vector will be multiplied by λ, which corresponds to a homothety. So the last result will have a norm of magnitude λ. We can write so:

$$\| \vec{x} \| = \lambda$$

And, in the same way:

$$\| \vec{y} \| = \mu$$

And so:

$$\| \vec{x} \| \| \vec{y} \| = \lambda \mu$$

On the other hand, as we have seen also in our previous paper (see [Torres-Heredia2025O]):

$$\vec{x} * \vec{y} = \lambda \mu R(\lambda + \theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So,

$$\| \vec{x} * \vec{y} \| = \lambda \mu$$

And finally we conclude that:

$$\| \vec{x} * \vec{y} \| = \| \vec{x} \| \| \vec{y} \|$$

We will use also the following inequality:

$$\| \vec{x} + \vec{y} \| \geq \| \vec{x} \| - \| \vec{y} \|$$

It comes from the fact that:

$$
\begin{aligned}
\| \vec{x} \| &= \| \vec{x} + \vec{y} - \vec{y} \| \\
&\leq \| \vec{x} + \vec{y} \| + \| \vec{y} \|
\end{aligned}
$$

So, we can write:

$$
\| \vec{x} + \vec{y} \| \geq \| \vec{x} \| - \| \vec{y} \|
$$

4 Limits

In our previous paper we have seen also the definition of a function in this framework (see [Torres-Heredia2025O]). Let $f(\vec{x})$ be a function defined in some deleted neighborhood of $\vec{x} = \vec{x_0}$. The vector $\vec{l}$ is the limit of $f(\vec{x})$ as $\vec{x}$ approaches $\vec{x_0}$, and we write $\lim_{\vec{x}\to\vec{x_0}} f(\vec{x}) = \vec{l}$, if and only if the following is true: for every $\varepsilon > 0$ there exists a number $\delta > 0$ with the property that $\| f(\vec{x}) - \vec{l} \| < \varepsilon$ for all values of $f(\vec{x})$ such that $\| \vec{x} - \vec{x_0} \| < \delta$ and $\vec{x} \neq \vec{x_0}$.

We are going to show an example which is an adaptation of what have been done until now with complex numbers (see [Spiegel2009], p. 61). Let $f(\vec{x}) = \vec{x}^2$. We are going to show that $\lim_{\vec{x}\to\begin{pmatrix}1\\2\end{pmatrix}} f(\vec{x}) = \begin{pmatrix}-3\\4\end{pmatrix}$.

Firstly, we are going to calculate $\begin{pmatrix}1\\2\end{pmatrix}^2$:

$$
\begin{aligned}
\begin{pmatrix}1\\2\end{pmatrix}^2 &= \begin{pmatrix}1\\2\end{pmatrix} * \begin{pmatrix}1\\2\end{pmatrix} \\
&= \begin{pmatrix}1\cdot 1 - 2\cdot 2\\1\cdot 2 + 2\cdot 1\end{pmatrix} \\
&= \begin{pmatrix}-3\\4\end{pmatrix}
\end{aligned}
$$

Now, we are going to show that, given any $\varepsilon > 0$, we can find a number $\delta > 0$ such that $\| \vec{x}^2 - \begin{pmatrix}-3\\4\end{pmatrix}^2 \| < \varepsilon$ whenever $0 < \| \vec{x} - \begin{pmatrix}-3\\4\end{pmatrix} \| < \delta$.

If $\delta \leqslant 1$, $0 < \| \vec{x} - \begin{pmatrix}-3\\4\end{pmatrix} \| < \delta$ implies that:

$$
\begin{aligned}
\| \vec{x}^2 - \begin{pmatrix}-3\\4\end{pmatrix}^2 \| &= \| (\vec{x} + \begin{pmatrix}-3\\4\end{pmatrix}) * (\vec{x} - \begin{pmatrix}-3\\4\end{pmatrix}) \| \\
&= \| \vec{x} - \begin{pmatrix}-3\\4\end{pmatrix} \| \| \vec{x} + \begin{pmatrix}-3\\4\end{pmatrix} \| \\
&< \delta \| \vec{x} + \begin{pmatrix}-3\\4\end{pmatrix} \| \\
&= \delta \| \vec{x} - \begin{pmatrix}-3\\4\end{pmatrix} + 2\begin{pmatrix}-3\\4\end{pmatrix} \| \\
&< \delta(\| \vec{x} - \begin{pmatrix}-3\\4\end{pmatrix} \| + \| 2\begin{pmatrix}-3\\4\end{pmatrix} \|) \\
&= \delta(\| \vec{x} - \begin{pmatrix}-3\\4\end{pmatrix} \| + 2 \| \begin{pmatrix}-3\\4\end{pmatrix} \|) \\
&< \delta(1 + 2 \| \begin{pmatrix}-3\\4\end{pmatrix} \|)
\end{aligned}
$$

We can take δ as 1 or $\dfrac{\varepsilon}{1 + 2 \| \begin{pmatrix}-3\\4\end{pmatrix} \|}$, the smallest of the two numbers. So we have $\| \vec{x}^2 - \begin{pmatrix}-3\\4\end{pmatrix}^2 \| < \varepsilon$ whenever $0 < \| \vec{x} - \begin{pmatrix}-3\\4\end{pmatrix} \| < \delta$.

For limits involving infinity, we can follow also an analogous way to what has been done with complex numbers. When the norm of a vector $\vec{x}$ approaches to infinity, we say that the vector $\vec{x}$ approaches to infinity.

Now, we are going to show that $\lim_{\|\vec{x}\| \to \infty} \| \vec{x}^n \| = \infty$, where n is natural number and $n \geqslant 1$.

In the previous section we have seen that:

$$\vec{x} = \lambda R(\theta) \begin{pmatrix}1\\0\end{pmatrix}$$

and that

$$\| \vec{x} \| = \lambda$$

So,

$$
\begin{aligned}
\| \vec{x}^n \| &= \| \vec{x} \|^n \\
&= \lambda^n
\end{aligned}
$$

So, it is clear that when $\| \vec{x} \|$ approaches to infinity, λ approaches to infinity, and also λ^n approaches to infinity. So we have proved that $\lim_{\|\vec{x}\| \to \infty} \| \vec{x}^n \| = \infty$.

Now, we are going to show that $\lim_{\|\vec{x}\| \to \infty} \| \dfrac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^n} \| = 0$.

We recall that $\begin{pmatrix}1\\0\end{pmatrix} = 1 \cdot R(0)\begin{pmatrix}1\\0\end{pmatrix}$, where R is a rotation matrix.

And we know, thanks to the definition of the division in our previous paper (see [Torres-Heredia2025O]), that:

$$\begin{aligned}\frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^n} &= \frac{1}{\lambda}R(0-\theta)\begin{pmatrix}1\\0\end{pmatrix}\\ &= \frac{1}{\lambda^n}R(-\theta)\begin{pmatrix}1\\0\end{pmatrix}\end{aligned}$$

So,

$$\| \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^n} \| = \frac{1}{\lambda^n}$$

So, it is clear that when $\| \vec{x} \|$ approaches to infinity, λ approaches to infinity, and $\frac{1}{\lambda^n}$ approaches to 0. So, we have proven that $\lim_{\|\vec{x}\|\to\infty} \| \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^n} \| = 0$.

Now, let's consider the following polynomial:

$$P(\vec{x}) = \vec{a_0} + \vec{a_1} * \vec{x} + \vec{a_2} * \vec{x}^2 + \vec{a_3} * \vec{x}^3 + \vec{a_4} * \vec{x}^4 + \cdots + \vec{a_n} * \vec{x}^n$$

We are going to show that $\lim_{\|\vec{x}\|\to\infty} \| P(\vec{x}) \| = \infty$.

Firstly,

$$\begin{aligned}\| P(\vec{x}) \| &= \| \vec{a_n} * \vec{x}^n * (\frac{\vec{a_0}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^n} + \frac{\vec{a_1}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^{n-1}} + \frac{\vec{a_2}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^{n-2}} + \frac{\vec{a_3}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^{n-3}} + \cdots + \begin{pmatrix}1\\0\end{pmatrix}) \|\\ &= \| \vec{a_n} * \vec{x}^n \| \, \| \frac{\vec{a_0}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^n} + \frac{\vec{a_1}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^{n-1}} + \frac{\vec{a_2}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^{n-2}} + \frac{\vec{a_3}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^{n-3}} + \cdots + \begin{pmatrix}1\\0\end{pmatrix} \|\\ &= \| \vec{a_n} * \vec{x}^n \| \, \| \begin{pmatrix}1\\0\end{pmatrix} + \frac{\vec{a_0}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^n} + \frac{\vec{a_1}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^{n-1}} + \frac{\vec{a_2}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^{n-2}} + \frac{\vec{a_3}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^{n-3}} + \cdots + \frac{\vec{a_{n-1}}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}} \|\end{aligned}$$

Now, we can write the following inequality involving one of those factors:

$$\| \begin{pmatrix}1\\0\end{pmatrix} + \frac{\vec{a_0}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^n} + \frac{\vec{a_1}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^{n-1}} + \cdots + \frac{\vec{a_{n-1}}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}} \| \geq \| \begin{pmatrix}1\\0\end{pmatrix} \| - \| \frac{\vec{a_0}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^n} + \frac{\vec{a_1}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}^{n-1}} + \cdots + \frac{\vec{a_{n-1}}}{\vec{a_n}} * \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\vec{x}} \|$$

On the other hand, we can write the following inequality involving the second term of the right side of the last inequality:

$$\| \frac{\vec{a_0}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}^n} + \frac{\vec{a_1}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}^{n-1}} + \cdots + \frac{\vec{a_{n-1}}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}} \| \leq \| \frac{\vec{a_0}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}^n} \| + \| \frac{\vec{a_1}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}^{n-1}} \| + \cdots + \| \frac{\vec{a_{n-1}}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}} \|$$

As we have seen before, the norm of the type $\| \vec{c} * \frac{\binom{1}{0}}{\vec{x}^k} \|$ will approach to 0 as $\| \vec{x} \|$ aproaches to infinity. In fact, in each norm, the fraction is multiplied by a constant vector and the norm of the result will approach also to 0.

So the right side of the last inequality will tend to 0. Then, we can find a number $M > 0$ such that, for $\| \vec{x} \| > M$ we have:

$$\| \frac{\vec{a_0}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}^n} \| + \| \frac{\vec{a_1}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}^{n-1}} \| + \cdots + \| \frac{\vec{a_{n-1}}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}} \| < \frac{1}{2} \| \binom{1}{0} \|$$

So, taking an inequality seen above, we can write:

$$\| \binom{1}{0} + \frac{\vec{a_0}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}^n} + \frac{\vec{a_1}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}^{n-1}} + \cdots + \frac{\vec{a_{n-1}}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}} \| \geq \| \binom{1}{0} \| - \frac{1}{2} \| \binom{1}{0} \|$$

And so,

$$\| \binom{1}{0} + \frac{\vec{a_0}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}^n} + \frac{\vec{a_1}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}^{n-1}} + \cdots + \frac{\vec{a_{n-1}}}{\vec{a_n}} * \frac{\binom{1}{0}}{\vec{x}} \| \geq \frac{1}{2} \| \binom{1}{0} \|$$

So, for $\| \vec{x} \| > M$, we have:

$$\begin{aligned} \| P(\vec{x}) \| &\geq \| \vec{a_n} * \vec{x}^n \| \frac{1}{2} \| \binom{1}{0} \| \\ &= \| \vec{a_n} * \vec{x}^n \| \frac{1}{2} \\ &= \frac{1}{2} \| \vec{a_n} \| \| \vec{x}^n \| \end{aligned}$$

And so, the right side of the inequality approaches to infinity as $\| \vec{x} \|$ aproaches to infinity. So we have proven that $\lim_{\|\vec{x}\| \to \infty} \| P(\vec{x}) \| = \infty$.

5 Basic derivatives

5.1 Derivatives of basic functions

In this framework we will define the derivative of a function f at $\vec{x_0}$ as this:

$$f'(\vec{x}) = \lim_{\Delta\vec{x} \to \vec{O}} \frac{f(\vec{x} + \Delta\vec{x}) - f(\vec{x})}{\Delta\vec{x}}$$

It is important to note the difference between this definition and the definition of the derivatives in Differential Geometry, where the derivatives of vector functions are calculated with respect to scalars. In this framework, the derivative of vector functions are calculated with respect to vectors.

For example,let $f(\vec{x}) = \vec{x}^2$. The derivative at $\vec{x_0}$ is:

$$\begin{aligned}
f'(\vec{x_0}) &= \lim_{\Delta\vec{x}\to\vec{O}} \frac{f(\vec{x_0}+\Delta\vec{x}) - f(\vec{x_0})}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{(\vec{x_0}+\Delta\vec{x})^2 - \vec{x_0}^2}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{\vec{x_0}^2 + 2\vec{x_0} * \Delta\vec{x} + (\Delta\vec{x})^2 - \vec{x_0}^2}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{2\vec{x_0} * \Delta\vec{x} + (\Delta\vec{x})^2}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} (2\vec{x_0} * \Delta\vec{x} + (\Delta\vec{x})^2) * (\Delta\vec{x})^{-1} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} 2\vec{x_0} * \Delta\vec{x} * (\Delta\vec{x})^{-1} + (\Delta\vec{x})^2 * (\Delta\vec{x})^{-1} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} 2\vec{x_0} + \Delta\vec{x} \\
&= 2\vec{x_0}
\end{aligned}$$

For the derivative of a constant, we will show firstly that the result of the multiplication of a vector $\vec{x}$ by $\vec{O}$ is $\vec{O}$:

$$\begin{aligned}
\vec{x} * \vec{O} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} * \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} x_1 * 0 - x_2 * 0 \\ x_1 * 0 + x_2 * 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&= \vec{O}
\end{aligned}$$

Now, let $f(\vec{x}) = \vec{c}$. The derivative at $\vec{x_0}$ is:

$$
\begin{aligned}
f'(\vec{x_0}) &= \lim_{\Delta\vec{x}\to\vec{O}} \frac{f(\vec{x_0}+\Delta\vec{x}) - f(\vec{x_0})}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{\vec{c}-\vec{c}}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{\vec{O}}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \vec{O} * (\Delta\vec{x})^{-1} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \vec{O} \\
&= \vec{O}
\end{aligned}
$$

For $f(\vec{x}) = \vec{x}$, the derivative at $\vec{x_0}$ is:

$$
\begin{aligned}
f'(\vec{x_0}) &= \lim_{\Delta\vec{x}\to\vec{O}} \frac{f(\vec{x_0}+\Delta\vec{x}) - f(\vec{x_0})}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{\vec{x_0}+\Delta\vec{x}-\vec{x_0}}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{\Delta\vec{x}}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned}
$$

Now, let $f(\vec{x}) = \vec{x}^n$, with $n \geq 2$. The derivative at $\vec{x_0}$ is:

$$
\begin{aligned}
f'(\vec{x_0}) &= \lim_{\Delta\vec{x}\to\vec{O}} \frac{f(\vec{x_0}+\Delta\vec{x}) - f(\vec{x_0})}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{(\vec{x_0}+\Delta\vec{x})^n - \vec{x_0}^n}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{C_0^n\vec{x_0}^n + C_1^n\vec{x_0}^{n-1} * \Delta\vec{x} + C_2^n\vec{x_0}^{n-2} * (\Delta\vec{x})^2 + \cdots + C_k^n\vec{x_0}^{n-k} * (\Delta\vec{x})^k + \cdots + C_n^n(\Delta\vec{x})^n - \vec{x_0}^n}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{C_1^n\vec{x_0}^{n-1} * \Delta\vec{x} + C_2^n\vec{x_0}^{n-2} * (\Delta\vec{x})^2 + \cdots + C_k^n\vec{x_0}^{n-k} * (\Delta\vec{x})^k + \cdots + C_n^n(\Delta\vec{x})^n}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} (C_1^n\vec{x_0}^{n-1} * \Delta\vec{x} + C_2^n\vec{x_0}^{n-2} * (\Delta\vec{x})^2 + \cdots + C_k^n\vec{x_0}^{n-k} * (\Delta\vec{x})^k + \cdots + C_n^n(\Delta\vec{x})^n) * (\Delta\vec{x})^{-1} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} C_1^n\vec{x_0}^{n-1} * \Delta\vec{x} * (\Delta\vec{x})^{-1} + \cdots + C_k^n\vec{x_0}^{n-k} * (\Delta\vec{x})^k * (\Delta\vec{x})^{-1} + \cdots + C_n^n(\Delta\vec{x})^n * (\Delta\vec{x})^{-1} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} C_1^n\vec{x_0}^{n-1} + C_2^n\vec{x_0}^{n-2} * \Delta\vec{x} + \cdots + C_k^n\vec{x_0}^{n-k} * (\Delta\vec{x})^{k-1} + \cdots + C_n^n(\Delta\vec{x})^{n-1} \\
&= n\vec{x_0}^{n-1}
\end{aligned}
$$

Now, let's see the constant multiple rule.
Let $f(\vec{x}) = \vec{c} * g(\vec{x})$. The derivative at $\vec{x_0}$ is:

$$
\begin{aligned}
f'(\vec{x_0}) &= \lim_{\Delta\vec{x}\to\vec{O}} \frac{\vec{c} * g(\vec{x_0}+\Delta\vec{x}) - \vec{c} * g(\vec{x_0})}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{\vec{c} * (g(\vec{x_0}+\Delta\vec{x}) - g(\vec{x_0}))}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \vec{c} * \frac{g(\vec{x_0}+\Delta\vec{x}) - g(\vec{x_0})}{\Delta\vec{x}} \\
&= \vec{c} * (\lim_{\Delta\vec{x}\to\vec{O}} \frac{g(\vec{x_0}+\Delta\vec{x}) - g(\vec{x_0})}{\Delta\vec{x}}) \\
&= \vec{c} * g'(\vec{x_0})
\end{aligned}
$$

For $f(\vec{x}) = g(\vec{x}) + h(\vec{x})$, the derivative at $\vec{x_0}$ is:

$$
\begin{aligned}
f'(\vec{x_0}) &= \lim_{\Delta\vec{x}\to\vec{O}} \frac{f(\vec{x_0}+\Delta\vec{x}) - f(\vec{x_0})}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{g(\vec{x_0}+\Delta\vec{x}) + h(\vec{x_0}+\Delta\vec{x}) - (g(\vec{x_0}) + h(\vec{x_0}))}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{g(\vec{x_0}+\Delta\vec{x}) + h(\vec{x_0}+\Delta\vec{x}) - g(\vec{x_0}) - h(\vec{x_0})}{\Delta\vec{x}} \\
&= \lim_{\Delta\vec{x}\to\vec{O}} \frac{g(\vec{x_0}+\Delta\vec{x}) - g(\vec{x_0})}{\Delta\vec{x}} + \lim_{\Delta\vec{x}\to\vec{O}} \frac{h(\vec{x_0}+\Delta\vec{x}) - h(\vec{x_0})}{\Delta\vec{x}} \\
&= g'(\vec{x_0}) + h'(\vec{x_0})
\end{aligned}
$$

5.2 The derivative of a polynomial

We will show an example. Let $P(\vec{x}) = \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix} * \vec{x} + \begin{pmatrix}2\\3\end{pmatrix} * \vec{x}^3 + \begin{pmatrix}1\\3\end{pmatrix} * \vec{x}^7 + \begin{pmatrix}4\\1\end{pmatrix} * \vec{x}^9$.

So,

$$\begin{aligned} P'(\vec{x}) &= \begin{pmatrix}2\\1\end{pmatrix} + 3\begin{pmatrix}2\\3\end{pmatrix} * \vec{x}^2 + 7\begin{pmatrix}1\\3\end{pmatrix} * \vec{x}^6 + 9\begin{pmatrix}4\\1\end{pmatrix} * \vec{x}^8 \\ &= \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}6\\9\end{pmatrix} * \vec{x}^6 + \begin{pmatrix}7\\21\end{pmatrix} * \vec{x}^6 + \begin{pmatrix}36\\9\end{pmatrix} * \vec{x}^8 \end{aligned}$$

6 The Taylor developments

6.1 Infinite series

So, with certain functions, we can develop infinite series by using the formula of Taylor:

$$f(\vec{x_0} + \vec{h}) = f(\vec{x_0}) + \vec{h} * f'(\vec{x_0}) + \frac{\vec{h}^2}{2!} f''(\vec{x_0}) + \cdots + \frac{\vec{h}^k}{k!} f^{(k)}(\vec{x_0}) + \cdots$$

and

$$f(\vec{x}) = f(\vec{x_0}) + f'(\vec{x_0}) * (\vec{x} - \vec{x_0}) + \frac{f''(\vec{x_0})}{2!} * (\vec{x} - \vec{x_0})^2 + \cdots + \frac{f^{(k)}(\vec{x_0})}{k!} * (\vec{x} - \vec{x_0})^k + \cdots$$

6.2 The use of a Taylor development in R as a scheme and the spiral paths

As we have seen in our first paper (see [Torres-Heredia2025A]), we can use the Taylor development of the real function e^x as scheme for a vector development.

We know that:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ... + \frac{1}{k!}x^k + ...$$

We worked with another function called Exp_v. If it is applied to the result of an operation X, we get:

$$Exp_v(X(\begin{pmatrix}1\\0\end{pmatrix})) = \begin{pmatrix}1\\0\end{pmatrix} + X(\begin{pmatrix}1\\0\end{pmatrix}) + \frac{1}{2!}X^2(\begin{pmatrix}1\\0\end{pmatrix}) + \frac{1}{3!}X^3(\begin{pmatrix}1\\0\end{pmatrix}) + ...$$

We applied that function to $H_\lambda \circ R(90°)(\begin{pmatrix}1\\0\end{pmatrix})$ and we got:

$$Exp_v(H_\lambda \circ R(90°)(\begin{pmatrix}1\\0\end{pmatrix})) = \begin{pmatrix}1\\0\end{pmatrix} + H_\lambda \circ R(90°)(\begin{pmatrix}1\\0\end{pmatrix}) + \frac{1}{2!}H_\lambda^2 \circ R^2(90°)(\begin{pmatrix}1\\0\end{pmatrix}) + \frac{1}{3!}H_\lambda^3 \circ R^3(90°)(\begin{pmatrix}1\\0\end{pmatrix}) + ...$$

And finally we got:

$$Exp_v(H_\lambda \circ R(90°)(\begin{pmatrix}1\\0\end{pmatrix})) = \cos(\lambda)\begin{pmatrix}1\\0\end{pmatrix} + \sin(\lambda)\begin{pmatrix}0\\1\end{pmatrix}$$

In fact, the partial sums of the Taylor development give us polynomials. And we represent graphically the paths of those partial sums, we get spirals which lead to a point on the unit circle, as we see in figure 1:

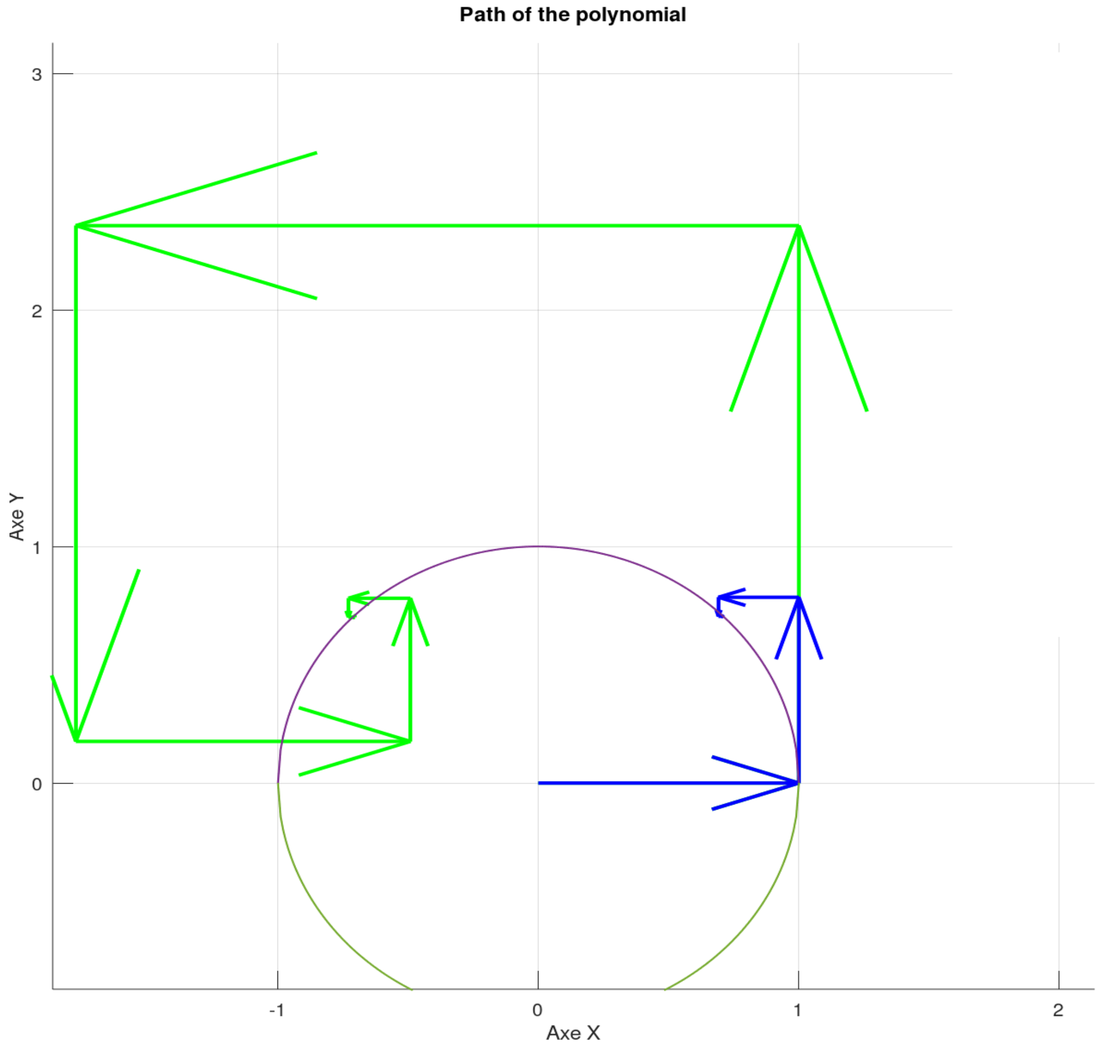

Figure 1: Representation of the spiral paths leading to points on the unit circle

The blue path corresponds to $\lambda = \dfrac{\pi}{4}$ and the green path corresponds to $\lambda = \dfrac{3\pi}{4}$

You will find a source code for GNU Octave in order to generate those paths (see [Torres-Heredia2025S]). In this code there is the implementation of some operations defined in our previous paper (see [Torres-Heredia2025O]).

It is important to note also that the mathematical objects we are studying can be expressed with matrices, as we have seen in our previous papers. That's why we can use also GNU Octave in order to work with matrices.

6.3 The Taylor development of a polynomial

We can get also the Taylor development of a polynomial. If we have the following polynomial:

$$P(\vec{x}) = \vec{a_0} + \vec{a_1} * \vec{x} + \vec{a_2} * \vec{x}^2 + \vec{a_3} * \vec{x}^3 + \vec{a_4} * \vec{x}^4 + \cdots + \vec{a_n} * \vec{x}^m$$

its Taylor development around $\vec{x_0}$ will be:

$$P(\vec{x}) = P(\vec{x_0}) + \vec{c_k} * (\vec{x} - \vec{x_0})^k + \vec{c_{k+1}} * (\vec{x} - \vec{x_0})^{k+1} + \cdots + \vec{c_m} * (\vec{x} - \vec{x_0})^m$$

We can show that there exists a sufficiently small number ρ such that, on the circle Γ centered at the final point of $\vec{x_0}$ with radius ρ, we have the following inequality:

$$\| \vec{c_{k+1}} * (\vec{x} - \vec{x_0})^{k+1} + \cdots + \vec{c_m} * (\vec{x} - \vec{x_0})^m \| < \| \vec{c_k} * (\vec{x} - \vec{x_0})^k \| = \| \vec{c_k} \| \rho^k$$

Firstly, because of the rules about the norm seen before, we know that:

$$\begin{aligned} \| \vec{c_k} * (\vec{x} - \vec{x_0})^k \| &= \| \vec{c_k} \| \| (\vec{x} - \vec{x_0})^k \| \\ &= \| \vec{c_k} \| \| (\vec{x} - \vec{x_0}) \|^k \\ &= \| \vec{c_k} \| \rho^k \end{aligned}$$

So, we want to show that:

$$\sum_{j=k+1}^{m} \| \vec{c_j} * (\vec{x} - \vec{x_0})^j \| < \| \vec{c_k} \| \rho^k$$

$$\begin{aligned} \sum_{j=k+1}^{m} \| \vec{c_j} * (\vec{x} - \vec{x_0})^j \| < \| \vec{c_k} \| \rho^k &\Leftrightarrow \sum_{j=k+1}^{m} \| \vec{c_j} \| \| (\vec{x} - \vec{x_0})^j \| < \| \vec{c_k} \| \rho^k \\ &\Leftrightarrow \sum_{j=k+1}^{m} \| \vec{c_j} \| \| (\vec{x} - \vec{x_0}) \|^j < \| \vec{c_k} \| \rho^k \\ &\Leftrightarrow \sum_{j=k+1}^{m} \| \vec{c_j} \| \rho^j < \| \vec{c_k} \| \rho^k \\ &\Leftrightarrow \sum_{j=1}^{m-k} \| \vec{c_j} \| \rho^j < \| \vec{c_k} \| \end{aligned}$$

Finally, ta sufficiently small number ρhe last inequality will be verified with a sufficiently small number ρ because $\vec{c_k}$ is a constant vector.

7 The Fundamental Theorem of Algebra

7.1 The formulation of the theorem in terms of paths

As we have seen in our first paper (see [Torres-Heredia2025A]), a polynomial $P(\vec{x})$ is a kind of program which, for each $\vec{x_0}$, allow us to go from the origin $(0, 0)$ to the final point of the vector $P(\vec{x_0})$.

Theorem 7.1. (Fundamental Theorem of Algebra) *For every non-constant polynomial of degree m having vectors as coefficients and vector variables, there are m vectors $\vec{r_k}$, $1 \leq k \leq m$, such that the path of the polynomial for each of those vectors $\vec{r_k}$ leads to the origin* $(0,0)$.

Obviously, this formulation is equivalent to the one with roots: every non-constant polynomial of degree m having vectors as coefficients, has m vector roots.

7.2 The proof

This is an adaptation of a proof given by Laurent Schwartz (see [Schwartz1995], pp. 215-216). There is also a quite similar proof, with the development of Taylor, in Wikipedia (see [WikipediaFit]).

We must to take into account the order of the roots. So if there is a root of the order 2, as we have seen in an example before, this root must be counted twice.

Actually, it is enough to prove that the polynomial has at least a root $\vec{x_0}$, for $m \geq 1$. Indeed, if we divide the polynomial by $\vec{x} - \vec{x_0}$, we get a polynomial of degree $m - 1$. And we can apply this reasoning to this new polynomial. And then we prove the theorem by recurrence on the degree of the polynomial.

So, let's consider the following polynomial:

$$P(\vec{x}) = \vec{a_0} + \vec{a_1} * \vec{x} + \vec{a_2} * \vec{x}^2 + \vec{a_3} * \vec{x}^3 + \vec{a_4} * \vec{x}^4 + \cdots + \vec{a_n} * \vec{x}^m$$

Let's suppose that it has no roots. So will show that there will be a contradiction,

We know that $\parallel P(\vec{x}) \parallel$, as we have seen before, tends towards infinity as $\parallel \vec{x} \parallel$ approaches infinity. So there is a number R such that, at the exterior of the circle of radius R centered at the point $(0,0)$ of $\mathbb{E}^2$, we have:

$$\parallel P(\vec{x}) \parallel \geq \parallel P(\begin{pmatrix}0\\0\end{pmatrix}) \parallel$$

We can consider $\parallel P \parallel$ as a function on the compact set defined by $\parallel \vec{x} \parallel \leq R$. So, because of the Extreme value theorem, $\parallel P \parallel$ attains a minimum $\mu > 0$. Let $\vec{x_0}$ a vector such that $\parallel P(\vec{x_0}) \parallel = \mu$. We have, for $\parallel \vec{x} \parallel \geq R$:

$$\parallel P(\vec{x}) \parallel \geq \parallel P(\begin{pmatrix}0\\0\end{pmatrix}) \parallel \geq \mu$$

So, the inequality $\parallel \vec{x} \parallel \geq \mu$ is verified for all $\vec{x}$. So μ is the minimum of the norm of P in $\mathbb{E}^2$. Now, let's consider the Taylor development of P around $\vec{x_0}$:

$$P(\vec{x}) = P(\vec{x_0}) + \vec{c_k} * (\vec{x} - \vec{x_0})^k + \vec{c_{k+1}} * (\vec{x} - \vec{x_0})^{k+1} + \cdots + \vec{c_m} * (\vec{x} - \vec{x_0})^m$$

We know that $\parallel P(\vec{x_0}) \parallel = \mu$. There exists a sufficiently small number ρ such that, on the circle Γ centered at the final point of $\vec{x_0}$ with radius ρ, we have the following inequality:

$$\parallel \vec{c_{k+1}} * (\vec{x} - \vec{x_0})^{k+1} + \cdots + \vec{c_m} * (\vec{x} - \vec{x_0})^m \parallel < \parallel \vec{c_k} * (\vec{x} - \vec{x_0})^k \parallel = \parallel \vec{c_k} \parallel \rho^k$$

We can suppose that ρ is such that $\parallel \vec{c_k} \parallel \rho^k < \mu$. Then, if the final point of $\vec{x}$ moves on the circle Γ, then the final point of $\vec{c_k} * (\vec{x} - \vec{x_0})^k$ moves on all the circle centered at the origin with radius $\parallel \vec{c_k} \parallel \rho^k < \mu$, and so the vector $P(\vec{x_0}) + \vec{c_k} * (\vec{x} - \vec{x_0})^k$ moves on all the circle centered at the final point of $P(\vec{x_0})$ with a radius of $\parallel \vec{c_k} \parallel \rho^k$, as so there is a vector $\vec{x_1}$ such that the final point of $P(\vec{x_0}) + \vec{c_k} * (\vec{x_1} - \vec{x_0})^k$ is on the segment between the origin and the final point of $P(\vec{x_0})$. Then we have:

$$\| P(\vec{x_0}) + \vec{c_k} * (\vec{x_1} - \vec{x_0})^k \| = \mu - \| \vec{c_k} \| \rho^k$$

So we have the following majoration:

$$\begin{aligned}
\| P(\vec{x_1}) \| &\leqslant \| P(\vec{x_0}) + \vec{c_k} * (\vec{x_1} - \vec{x_0})^k \| + \| \vec{c_{k+1}} * (\vec{x} - \vec{x_0})^{k+1} + \cdots + \vec{c_m} * (\vec{x} - \vec{x_0})^m \| \\
&< (\mu - \| \vec{c_k} \| \rho^k) + \| \vec{c_k} \| \rho^k \\
&= \mu
\end{aligned}$$

But the inequality $\| P(\vec{x_1}) \| < \mu$ contradicts the definition of $\vec{x_0}$.

8 The algorithm of Kneser

In 1940, H. Kneser (see [Kneser1940]) developed an intuitionistic proof in order to find a root of the polynomial (see also [RahmanSchmeisser2002], p. 63). His son M. Kneser improved the proof (see [Kneser1981]).

This is an adaptation of the classsical algorithm of M. Kneser as it is briefly repeated in the introduction of a paper before the authors present a more detailed version and improve later that method (see [GeuversWiedijkZwanenburg2001]). It is a constructive version of the simple proof which leads to a contradiction from the fact that the polynomial P is minimal at $\vec{x_0}$ with $\| P(\vec{x_0}) \| \neq 0$. As in the previous section, we begin with a non-constant polynomial:

$$P(\vec{x}) = \vec{a_0} + \vec{a_1} * \vec{x} + \vec{a_2} * \vec{x}^2 + \vec{a_3} * \vec{x}^3 + \vec{a_4} * \vec{x}^4 + \cdots + \vec{a_n} * \vec{x}^m$$

As we have seen before, P must have a minimum. This time we will assume that the minimum is reached for $\vec{x} = \vec{O}$. If the minimum is reached for another vector $\vec{x_0}$, then we take the polynomial $Q(\vec{x}) = P(\vec{x} + \vec{x_0})$. Now, let's assume that the minimum of $\| P(\vec{x}) \|$ is not 0. For reasons similar to what we have seen in the previous section, we can write:

$$P(\vec{x}) = \vec{c_0} + \vec{c_k} * \vec{x}^k + O(\vec{x}^{k+1})$$

where $\vec{c_k} \neq \vec{O}$. So $P(\vec{O}) = \vec{c_0} \neq \vec{O}$. So we can take:

$$\vec{x} = \varepsilon \sqrt[k]{-\frac{\vec{c_0}}{\vec{c_k}}}$$

where $\varepsilon > 0$. And, if ε is small enough, the part $O(\vec{x}^{k+1})$ will be negligible in comparison to the rest. So we are going to get a vector $\vec{x} \neq \vec{O}$ such that:

$$
\begin{aligned}
\| P(\vec{x}) \| &= \| \vec{c_0} + \vec{c_k} * (\varepsilon \sqrt[k]{-\frac{\vec{c_0}}{\vec{c_k}}})^k \| \\
&= \| \vec{c_0} + \vec{c_k} * (\varepsilon^k (-\frac{\vec{c_0}}{\vec{c_k}}) \| \\
&= \| \vec{c_0} + (\varepsilon^k (-\frac{\vec{c_0} * \vec{c_k}}{\vec{c_k}}) \| \\
&= \| \vec{c_0} - \varepsilon^k \vec{c_0} \| \\
&= \| \vec{c_0}(1 - \varepsilon^k) \| \\
&= | 1 - \varepsilon^k | \| \vec{c_0} \| \\
&= (1 - \varepsilon^k) \| \vec{c_0} \| \\
&< \| P(\vec{O}) \|
\end{aligned}
$$

So $\| P(\vec{O}) \|$ is not a minimum and it contradicts what was supposed before.

And, actually we have found a vector which give us a path which finishes closer to the point $(0,0)$. So we can repeat that process which will give a Cauchy sequence and finally we will find the solution which gives the path finishing at the point $(0,0)$.

As we read in (see [GeuversWiedijkZwanenburg2001]), this approach could be difficult because the choice of ε. ε should not be too small in order to reach the solution in a reasonable amount of time (and the steps should be countable), and should be quite small in order for the part $O(\vec{x}^{k+1})$ to be negligible.

In order to improve that, instead of the representation in which $\vec{c_k} * \vec{x}^k$ is the smallest power with a coefficient different from $\vec{O}$, we will take some appropriate k, non necessarily the smallest, and write $P(\vec{x})$ as:

$$P(\vec{x}) = \vec{c_0} + \vec{c_k} * (\vec{x} - \vec{x_0})^k + \textit{the other terms}$$

If we follow that way, $\| P(\vec{x}) \| < \| P(\vec{O}) \|$ and even $\| P(\vec{x}) \| < q \| P(\vec{O}) \|$ for some fixed $q < 1$.

9 Conclusion

So we have shown that the Fundamental Theorem of Algebra has an intrinsic geometrical nature. We have shown that it is not a theorem of Algebra because, as we have seen in our previous papers, the polynomials studied are actually geometrical. We have proved that theorem with vectors and topological tools.

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