A relationship between Golden Ratio and the length of the arc of the parabola $y = x^2$

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Abstract

This paper presents a simple relationship between Golden Ratio ϕ and the length of the arc from $x = 0$ to $x = 1$ of the curve defined by the equation $y = x^2$.

1 Introduction

The Golden Ratio is defined by the following expression:

$$\phi = \frac{1 + \sqrt{5}}{2}$$

It appears in geometry, architecture, etc. and has been studied for the last two millennials.

We know also some important properties of the Golden Ratio which will be useful (see [Corb13]) :

$$\frac{1}{\phi} = \phi - 1$$

$$\phi^3 = 2\phi + 1$$

In the following sections we are going to see that it appears also in the expression of the length of the arc from $x = 0$ to $x = 1$ of the curve defined by the equation $y = x^2$.

2 The calculation of the arc of the curve

We are interested in a curve defined by the equation $y = f(x)$ where $f(x) = x^2$. In that case, the arc length from $x = a$ to $x = b$ is calculated by the formula (see [Comm02]):

$$l = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

As we are interested in the length of that curve between $x = 0$ and $x = 1$, we have:

$$f'(x) = 2x \quad \Rightarrow \quad (f'(x))^2 = 4x^2$$

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$$\Rightarrow l = \int_0^1 \sqrt{1+4x^2}\,dx$$

We can evaluate it thanks to a hyperbolic substitution:

$$x = \frac{1}{2}\sinh(t) \quad \Rightarrow \quad dx = \frac{1}{2}\cosh(t)\,dt$$

By using the computer algebra system Maxima, we get:

$$\frac{\operatorname{asinh} 2 + 2\sqrt{5}}{4}$$

So:

$$l = \frac{1}{4}\sinh^{-1}(2) + \frac{1}{2}\sqrt{5}$$

We use the following definition:

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$$

So we get:

$$l = \frac{1}{4}(\ln(2 + \sqrt{2^2+1}) + 2\sqrt{5})$$

And finally:

$$\boxed{l = \frac{1}{4}(2\sqrt{5} + \ln(2+\sqrt{5}))}$$

By using the computer algebra system Maxima, we get a numerical approximation:

$$l \approx 1.478942857544598$$

3 Showing the Golden Ratio

We recall that the length of the arc from $x = 0$ to $x = 1$ of the curve defined by the equation $y = x^2$ is:

$$l = \frac{1}{4}(2\sqrt{5} + \ln(2+\sqrt{5}))$$

We rewrite this expression in the following forms:

$$l = \frac{1}{4}(2(\sqrt{5}+1-1) + \ln(1+1+\sqrt{5}))$$

$$l = \frac{1}{4}(2(2\frac{1+\sqrt{5}}{2} - 1) + \ln(1 + 2\frac{1+\sqrt{5}}{2}))$$

We recognize the Golden Ratio, so we write:

$$l = \frac{1}{4}(2(2\phi - 1) + \ln(1+2\phi))$$

We rewrite this expression in the following form:

$$l = \frac{1}{4}(2(\phi + \phi - 1) + \ln(2\phi + 1))$$

And now, by using the properties of the Golden Ratio seen above, we get:

$$l = \frac{1}{4}(2(\phi + \frac{1}{\phi}) + \ln(\phi^3))$$

So:

$$\boxed{l = \frac{1}{4}(2\phi + \frac{2}{\phi} + 3\ln(\phi))}$$

4 Conclusion

So we have found a simple relationship between Golden Ratio and the length of the arc from $x = 0$ to $x = 1$ of the curve defined by the equation $y = x^2$. This relationship is expressed by a simple formula involving ϕ and $ln(\phi)$.

References

[Corb13] F. Corbalán, *Le nombre d'or - Le langage mathématique de la beauté*, RBA France, SARL, 2013.

[Comm02] Comissions romandes de mathématiques, de physique et de chimie *Formulaires et tables - Mathématiques Physique Chimie*, Editions du Tricorne, 2002.